Divide-and-Conquer Checkpointing for Arbitrary Programs with No User Annotation

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The Future of Gradient-Based Machine Learning Software & Techniques
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Joint work with Barak Avrum Pearlmutter

Barak and my Work

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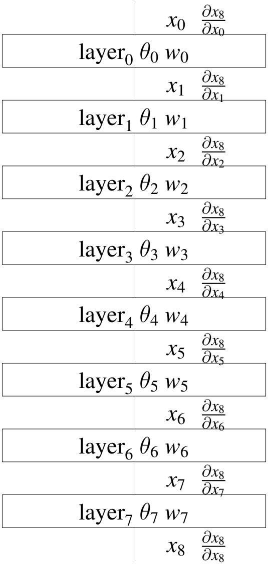
AD in functional programs.

Barak and my Work

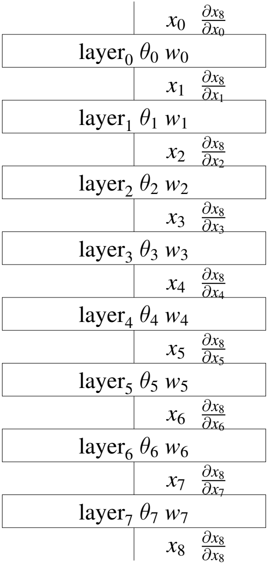
AD in functional programs.

AD is easier in functional programs.

A Neural Network



A Neural Network is a (Functional) Program



$$
\begin{array}{l}
\text{net}\ [\theta_0,\ldots,\theta_7]\ [w_0,\ldots,w_7]\ x_0 \triangleq \\
\quad \textbf{let}\ \begin{array}{rcl}
x_1 & = & \text{layer}_0\ \theta_0\ w_0\ x_0 \\
x_2 & = & \text{layer}_1\ \theta_1\ w_1\ x_1 \\
x_3 & = & \text{layer}_2\ \theta_2\ w_2\ x_2 \\
x_4 & = & \text{layer}_3\ \theta_3\ w_3\ x_3 \\
x_5 & = & \text{layer}_4\ \theta_4\ w_4\ x_4 \\
x_6 & = & \text{layer}_5\ \theta_5\ w_5\ x_5 \\
x_7 & = & \text{layer}_6\ \theta_6\ w_6\ x_6 \\
x_8 & = & \text{layer}_7\ \theta_7\ w_7\ x_7
\end{array} \\
\quad \textbf{in}\ x_8
\end{array}
$$

A (Functional) Program

$$
\begin{aligned}
& f\ [w_0, w_1]\ [x_0, x_1] \triangleq \\
& \quad \textbf{let}\ \ t_0 = w_0 \times x_0 \\
& \qquad\quad\ t_1 = w_1 \times x_1 \\
& \qquad\quad\ \ y = t_0 + t_1 \\
& \quad \textbf{in}\ y
\end{aligned}
$$

A (Functional) Program is a (Neural) Network

$$
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\quad \textbf{let}\ \begin{array}{rcl} t_0 & = & w_0 \times x_0 \\ t_1 & = & w_1 \times x_1 \\ y & = & t_0 + t_1 \end{array} \\
\quad \textbf{in}\ y
\end{array}
$$

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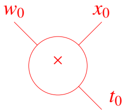
\end{aligned}$$

$$\textbf{in}\ y$$



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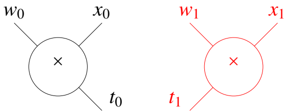

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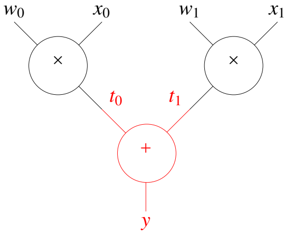

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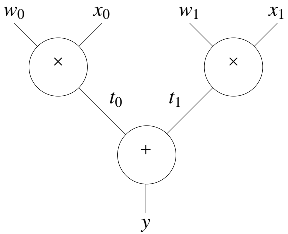

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- Deep learning network 'frameworks' are domain specific (functional) programming languages.
- A deep neural network is a long running (functional) program.
- Can perform backpropagation on (functional) programs by having an execution of the program generate a network. This is called reverse-mode automatic differentiation (AD).

A (Brief) History of Backpropagation aka Reverse-Mode AD

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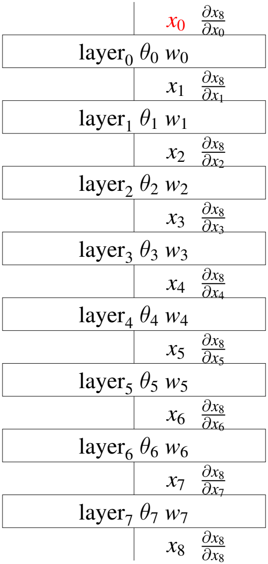
A.E. Bryson, Jr. and Y.-C. Ho, *Applied optimal control*, Blaisdell, 1969.

Evaluating a Neural Network



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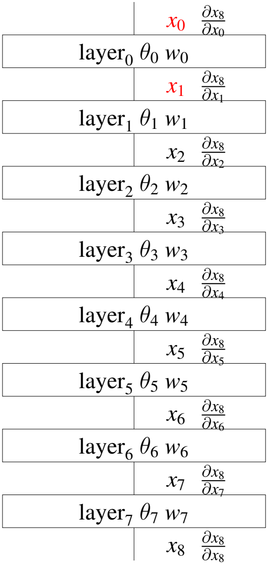

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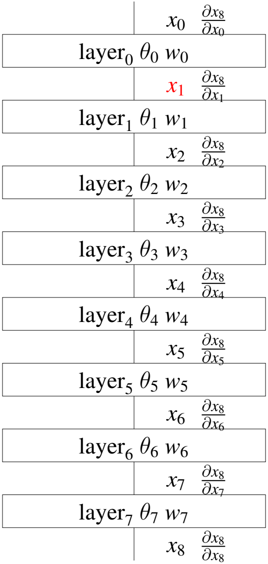

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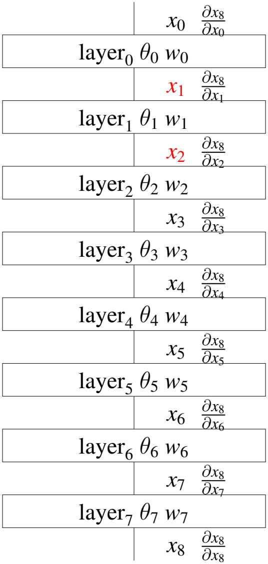

Evaluating a Neural Network

x_0 $\frac{\partial x_8}{\partial x_0}$

layer$_0$ θ_0 w_0

x_1 $\frac{\partial x_8}{\partial x_1}$

layer$_1$ θ_1 w_1

$\color{red}{x_2}$ $\frac{\partial x_8}{\partial x_2}$

layer$_2$ θ_2 w_2

x_3 $\frac{\partial x_8}{\partial x_3}$

layer$_3$ θ_3 w_3

x_4 $\frac{\partial x_8}{\partial x_4}$

layer$_4$ θ_4 w_4

x_5 $\frac{\partial x_8}{\partial x_5}$

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layer$_7$ θ_7 w_7

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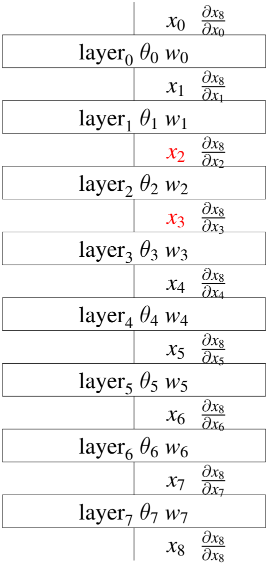

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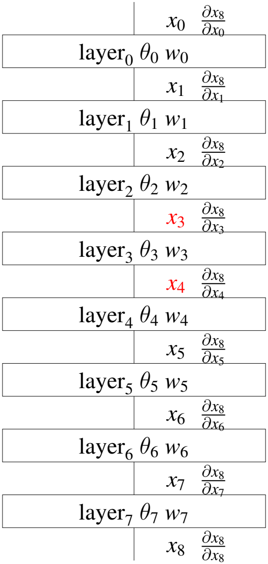

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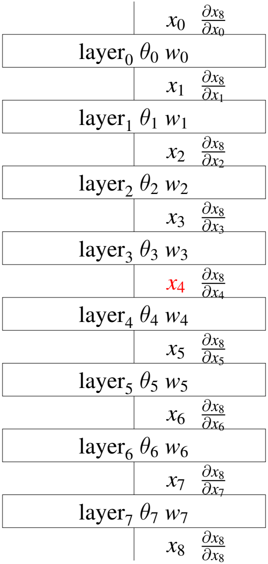

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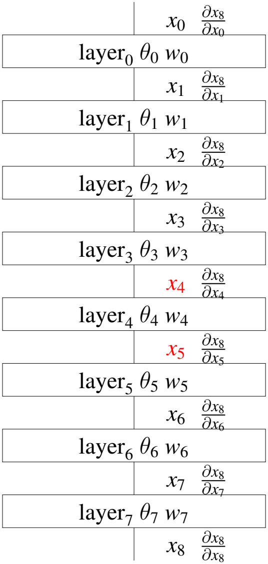

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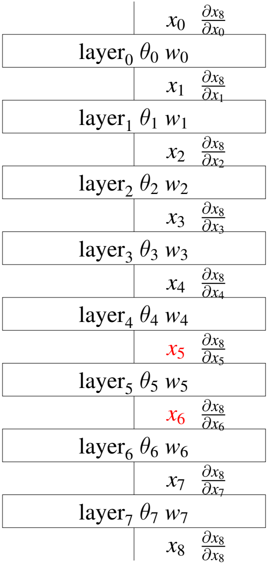

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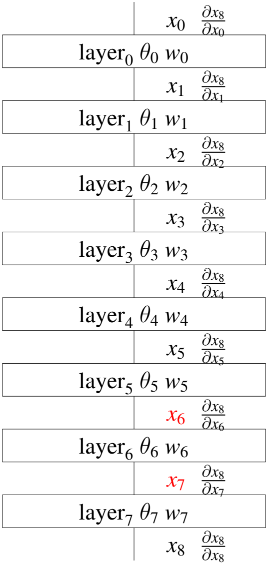

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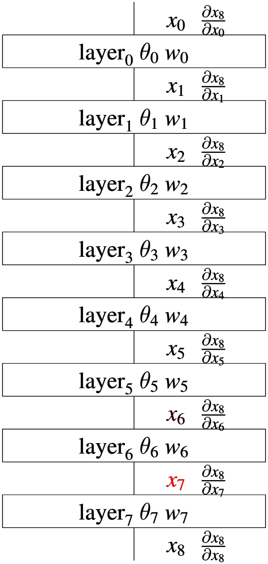

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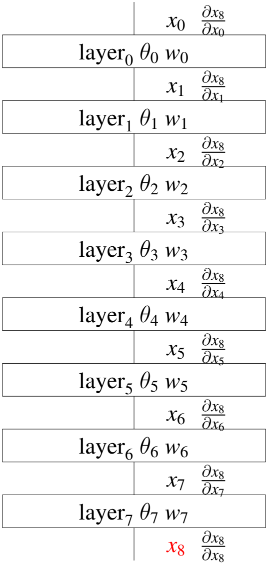

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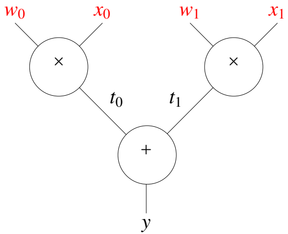

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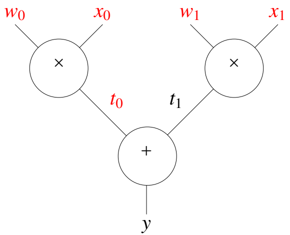

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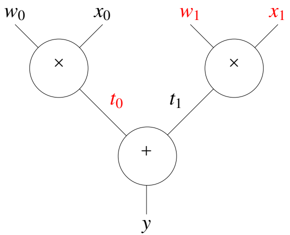

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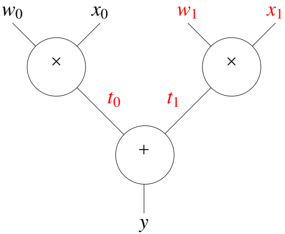

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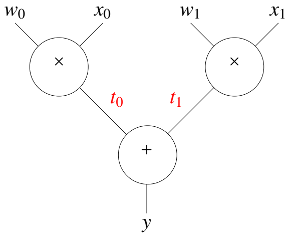

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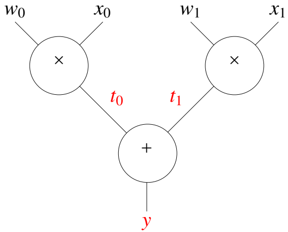

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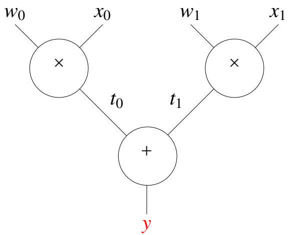

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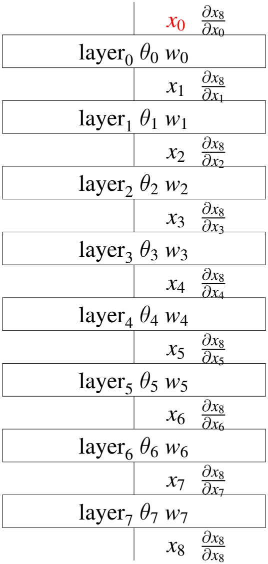
- Most deep learning frameworks store all intermediate variables to allow subsequent backpropagation.

Evaluating a Neural Network to Allow Subsequent Backpropagation



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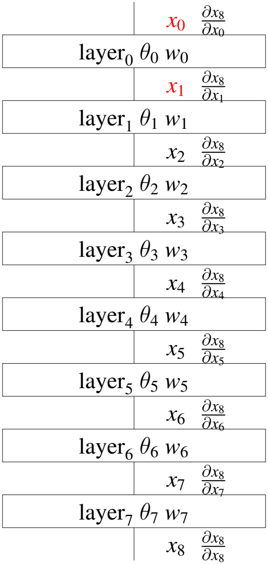

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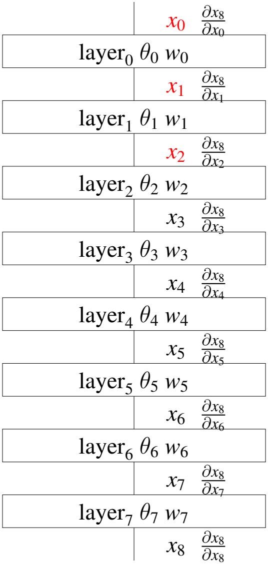

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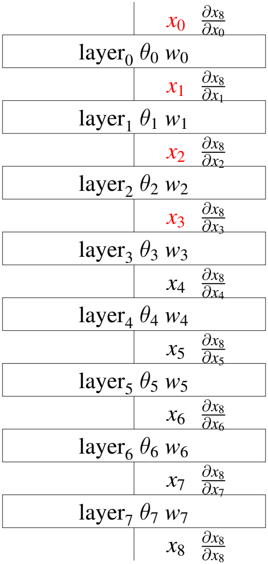

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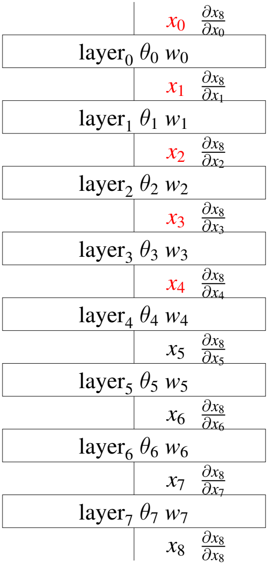

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x_1 $\frac{\partial x_8}{\partial x_1}$

layer$_1$ θ_1 w_1

x_2 $\frac{\partial x_8}{\partial x_2}$

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x_4 $\frac{\partial x_8}{\partial x_4}$

layer$_4$ θ_4 w_4

x_5 $\frac{\partial x_8}{\partial x_5}$

layer$_5$ θ_5 w_5

x_6 $\frac{\partial x_8}{\partial x_6}$

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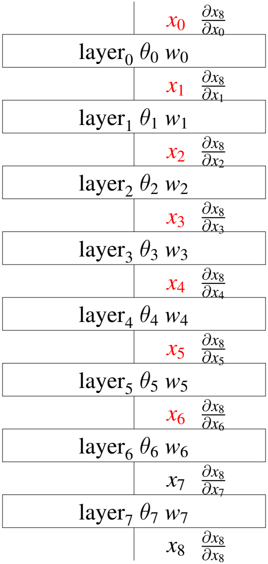

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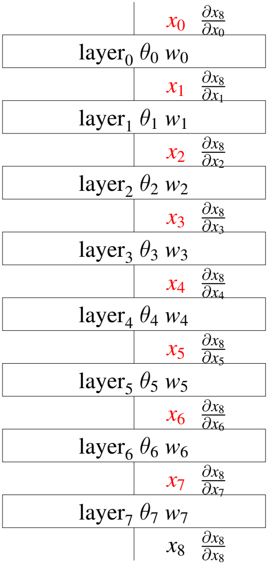

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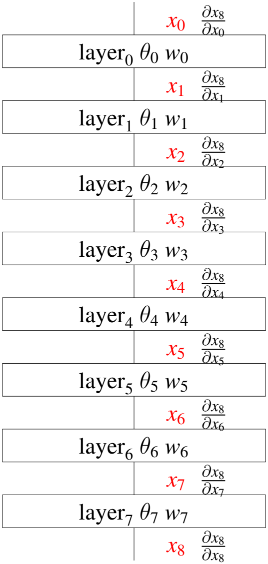

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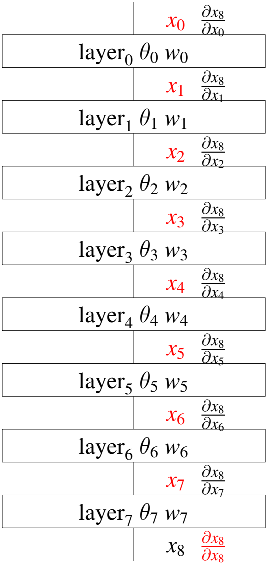

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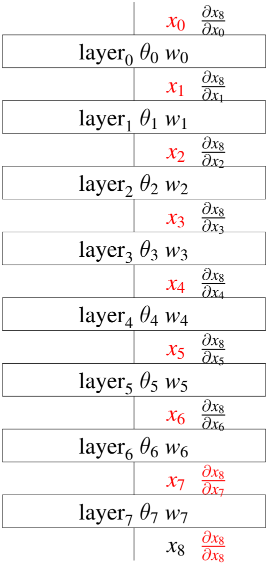
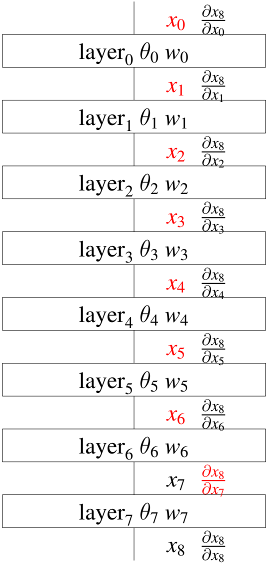


Evaluating a Neural Network to Allow Subsequent Backpropagation



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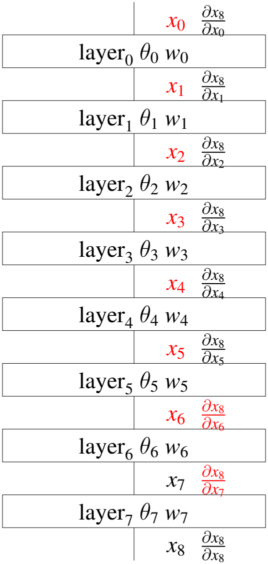

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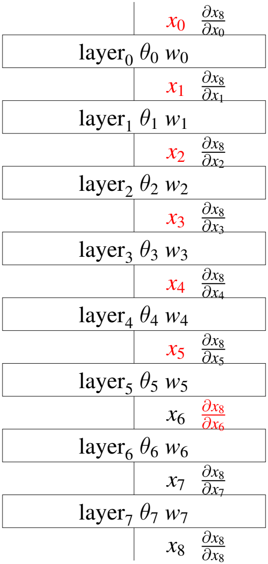

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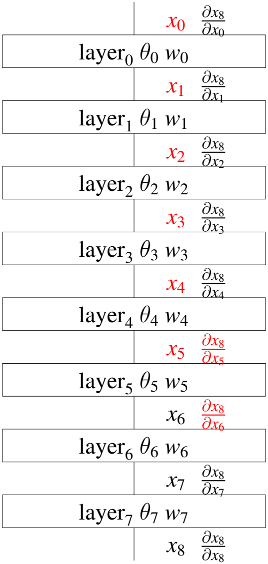

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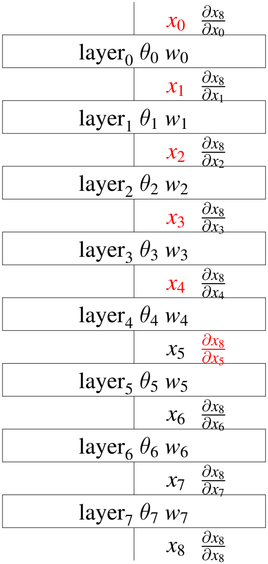

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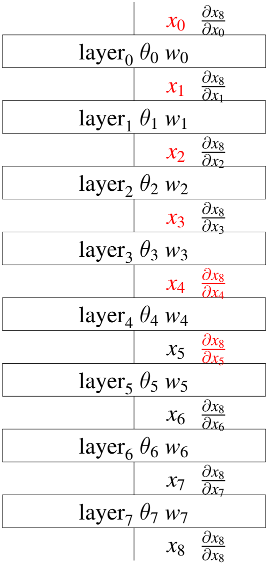

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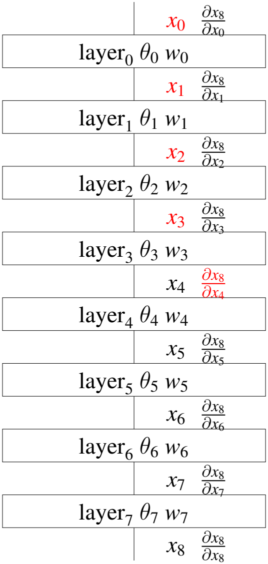

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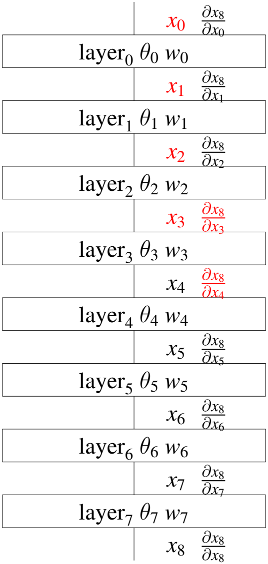

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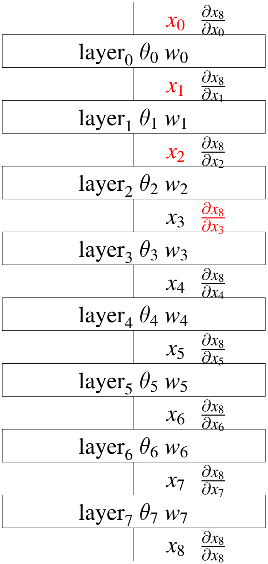

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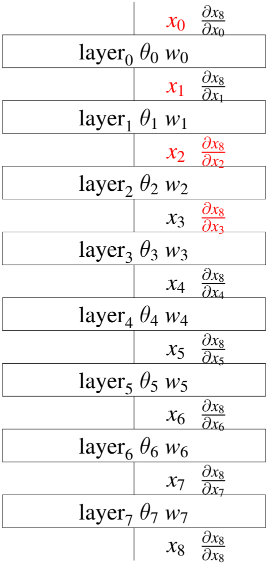

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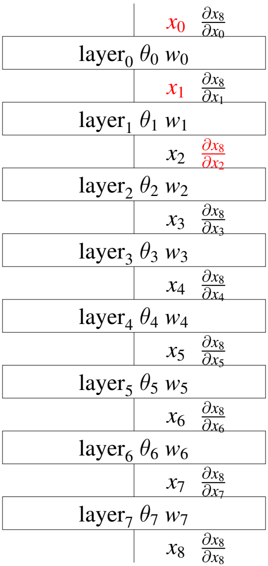

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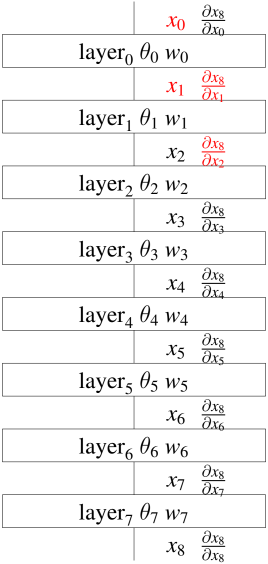

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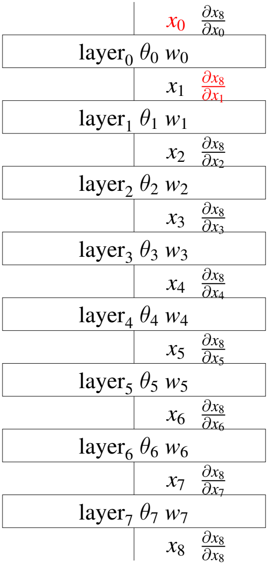

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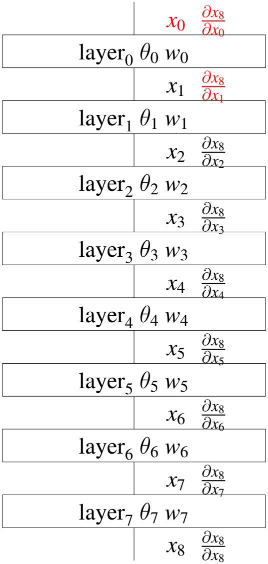

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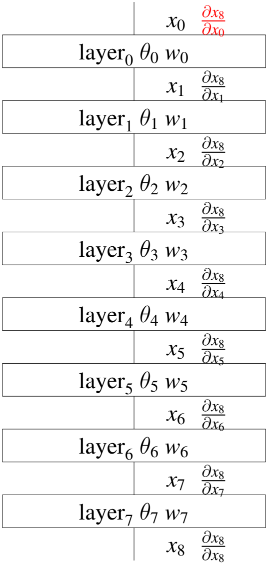

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Complexity of Reverse-Mode AD

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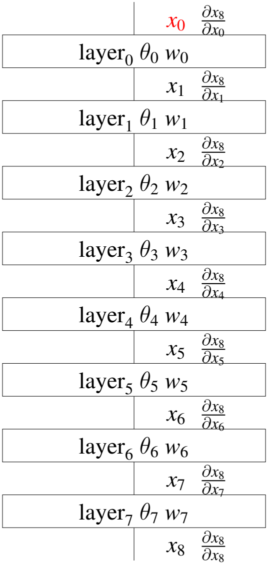

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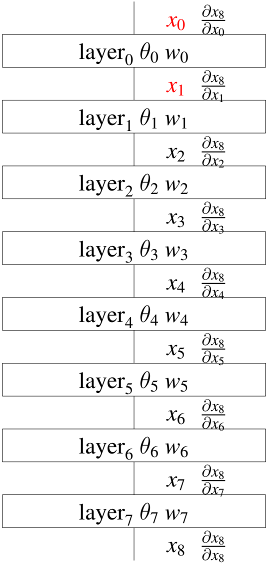

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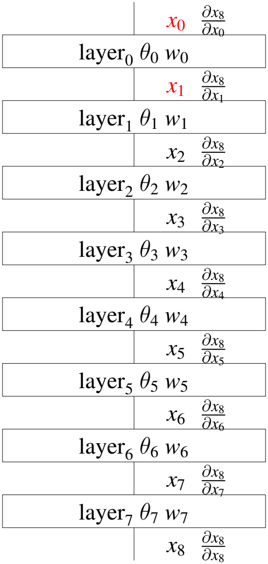

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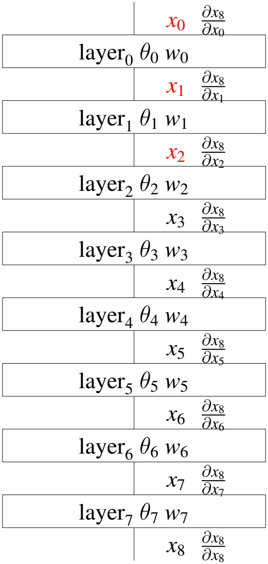

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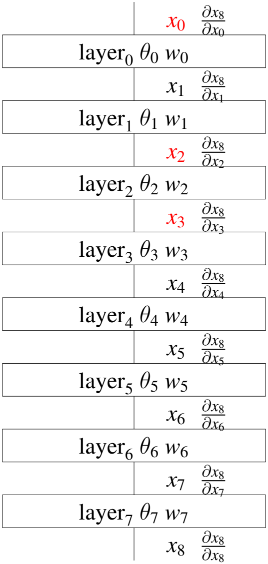

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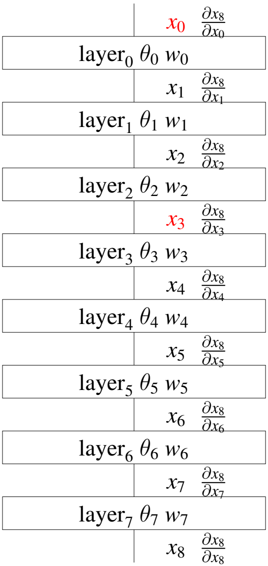

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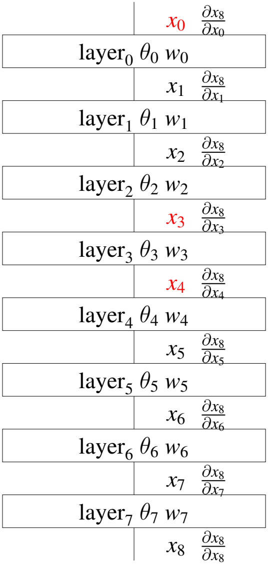

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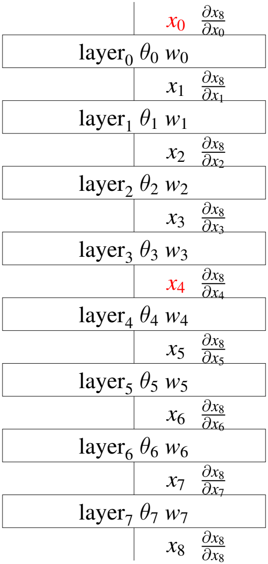

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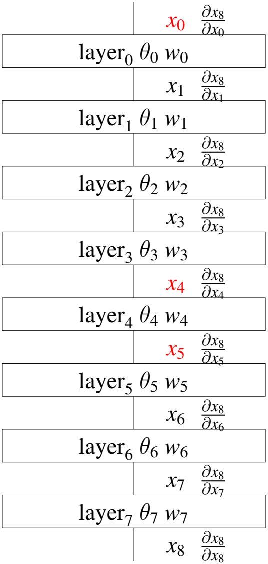

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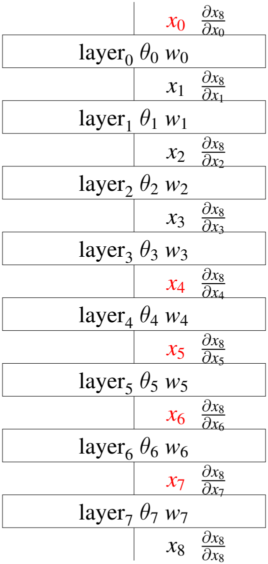

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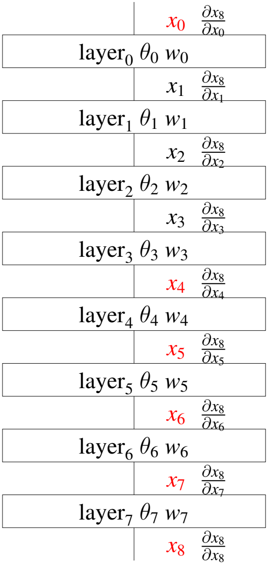

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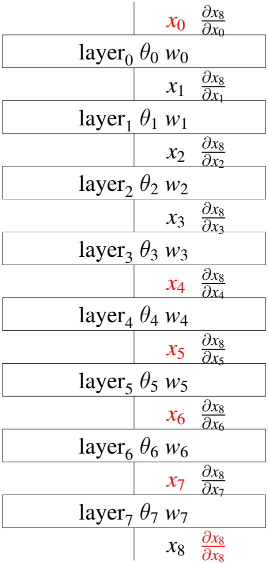

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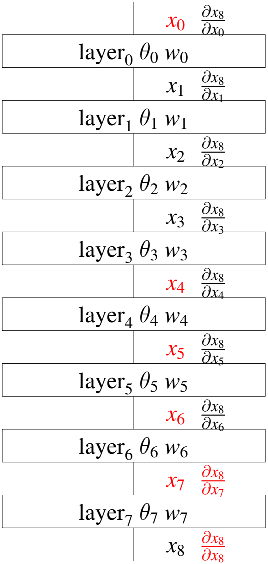

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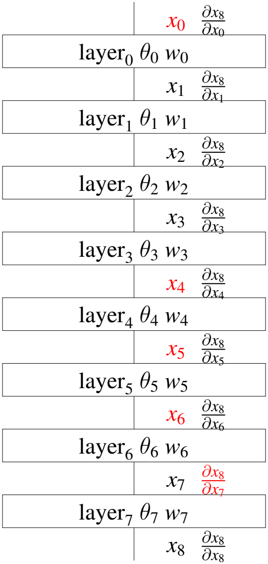

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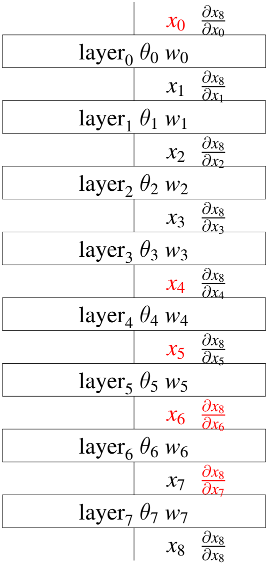

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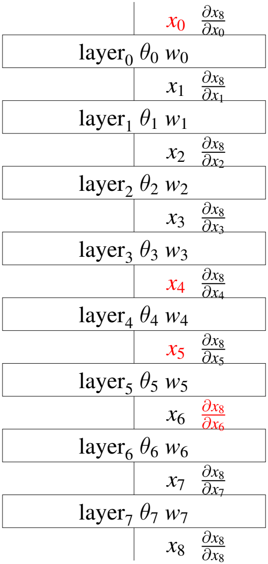

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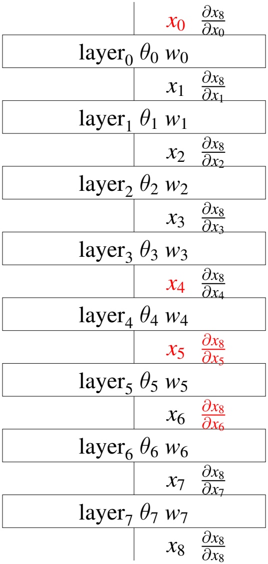

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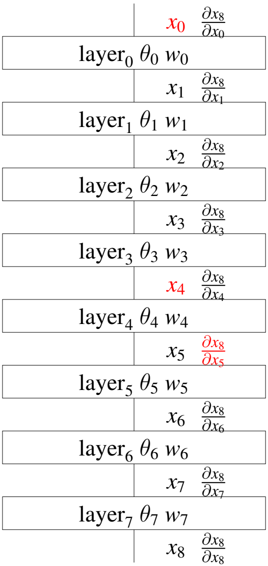

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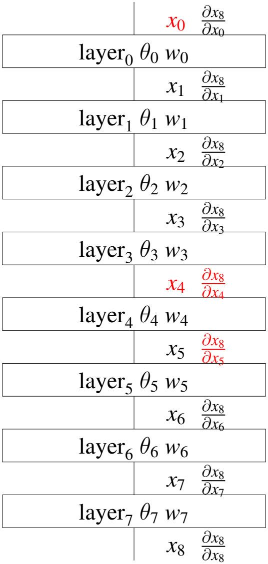

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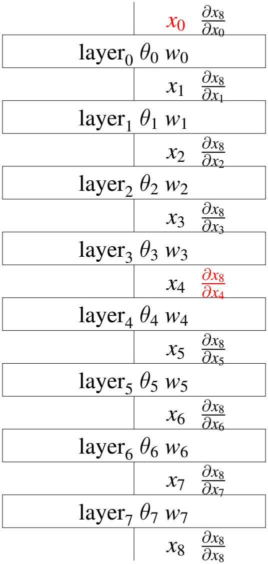

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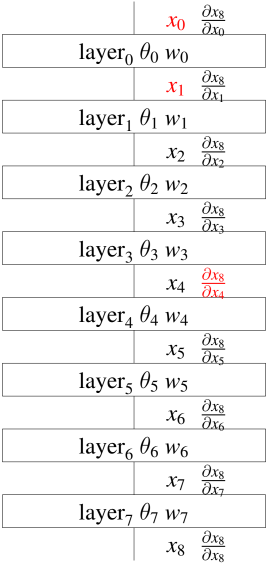

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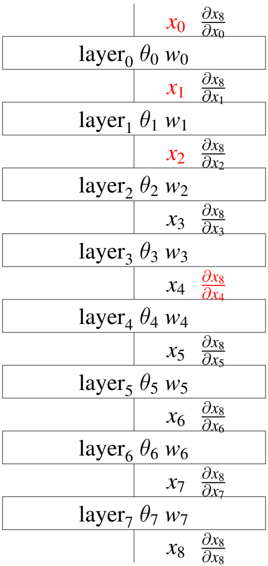

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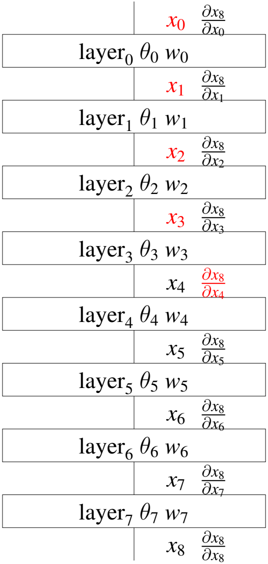

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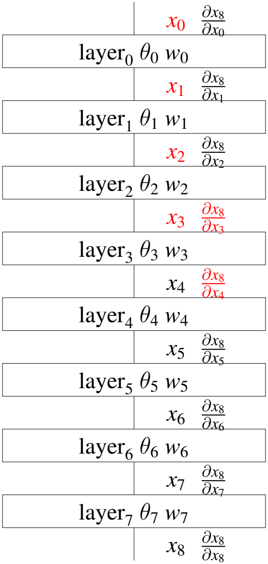

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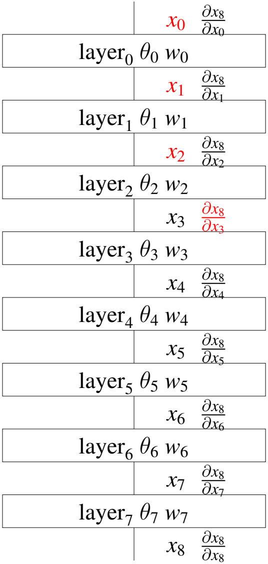

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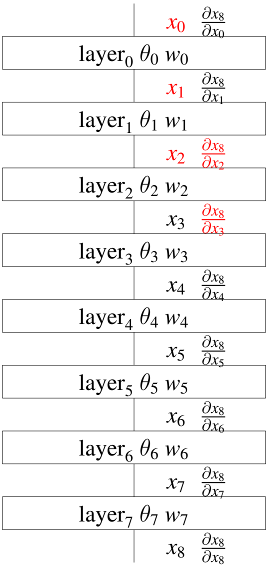

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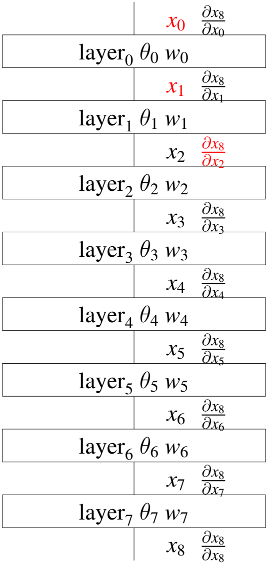

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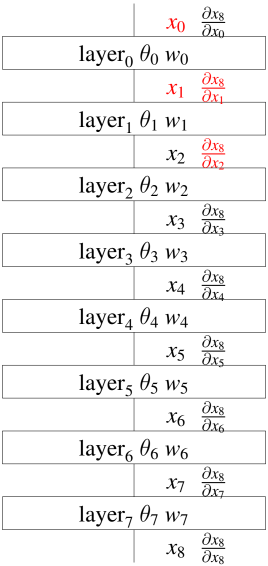

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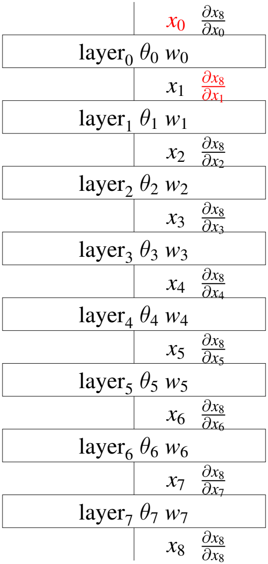

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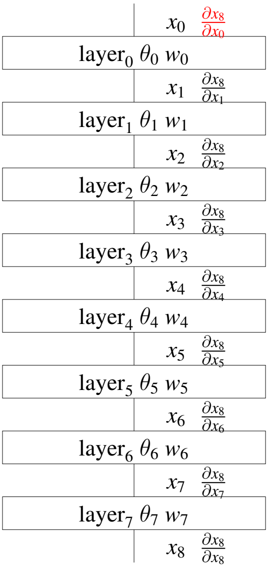

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- Can perform divide-and-conquer.

Divide-and-Conquer Checkpointing

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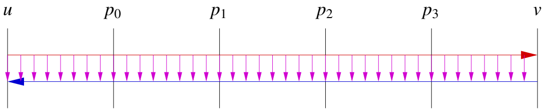

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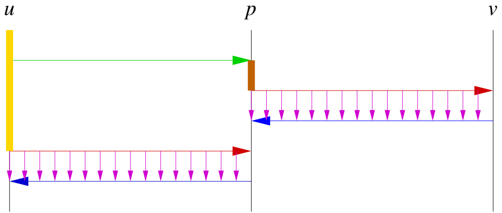

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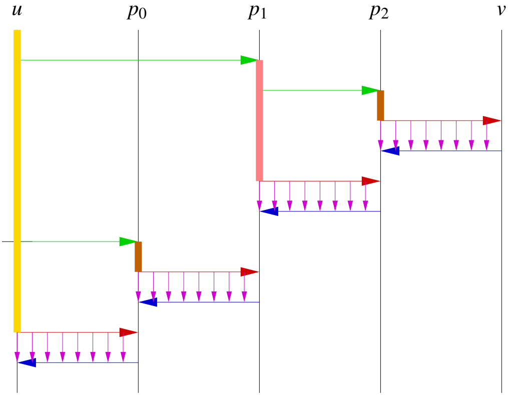

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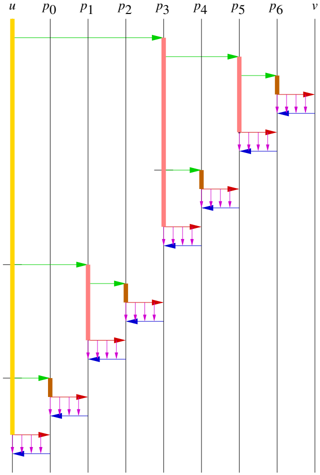

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A (Brief) History of Divide-and-Conquer Checkpointing

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A. Gruslys, R. Munos, I. Danihelka, M. Lanctot, and A. Graves, *Memory-Efficient Backpropagation Through Time*, NIPS, 2016.

A. Griewank, *Achieving Logarithmic Growth of Temporal and Spatial Complexity in Reverse Automatic Differentiation*, Optimization Methods and Software, 1:35-54, 1992.

Implemented for DO Loops

L. Hascoët and V. Pascual, *TAPENADE 2.1 User's Guide*, Rapport technique 300, INRIA, 2004.

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      do 10 i=1, n
         ...
10    continue
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      do 10 i=1, n            c$ad binomial-ckp n+1 30 1
         ...             ⇝          do 10 i=1, n
10    continue                          ...
                              10    continue
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https://www-sop.inria.fr/tropics/tapenade/faq.html

> *Assuming that the final number of iterations N is known, and assuming that each iteration has the same runtime cost,*

Desiderata

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A (deep) neural network has no loops (except inside primitives).

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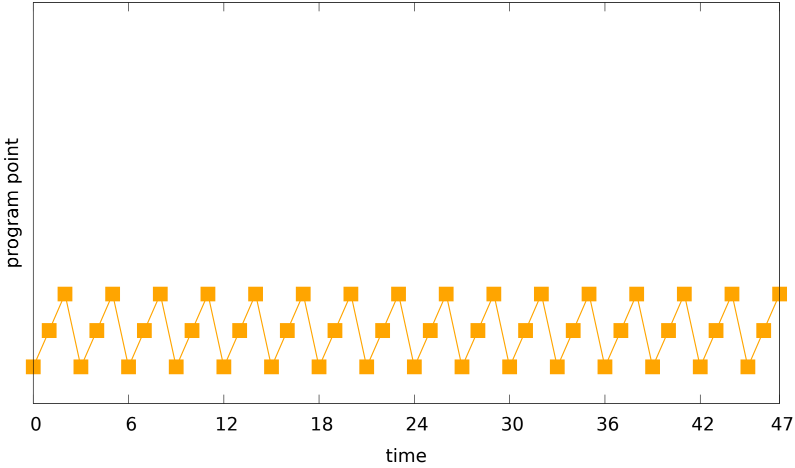
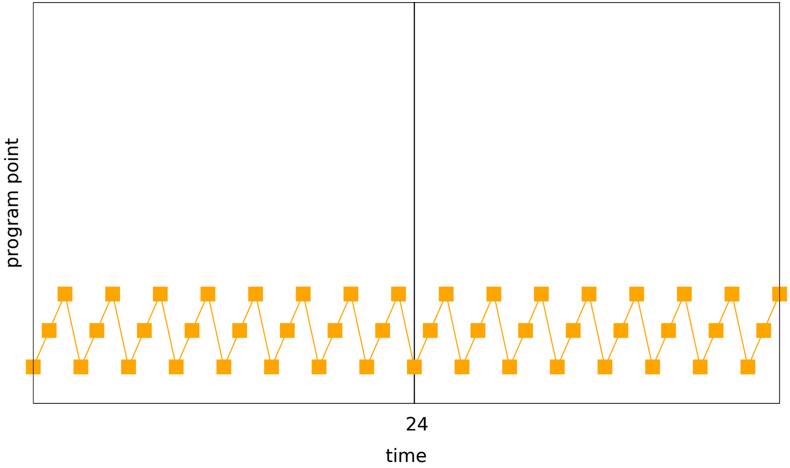
Want to implement for arbitrary code (not just a single DO loop).

Execution Trace of Loop



Execution Trace of Loop

Easy to make regular and uniform checkpoints



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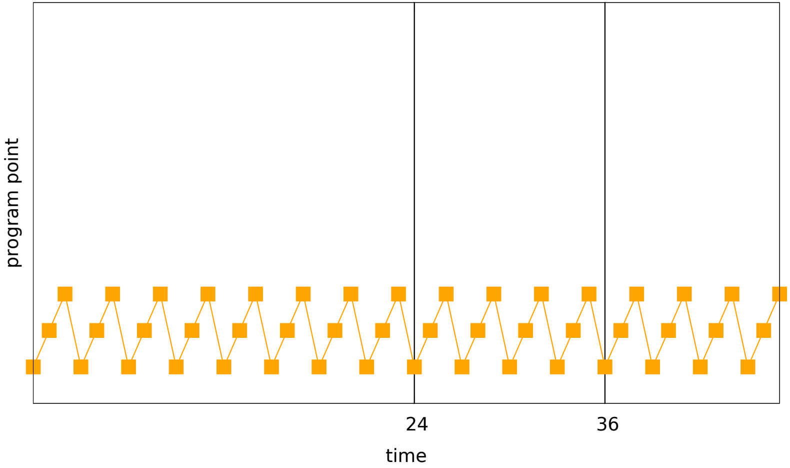

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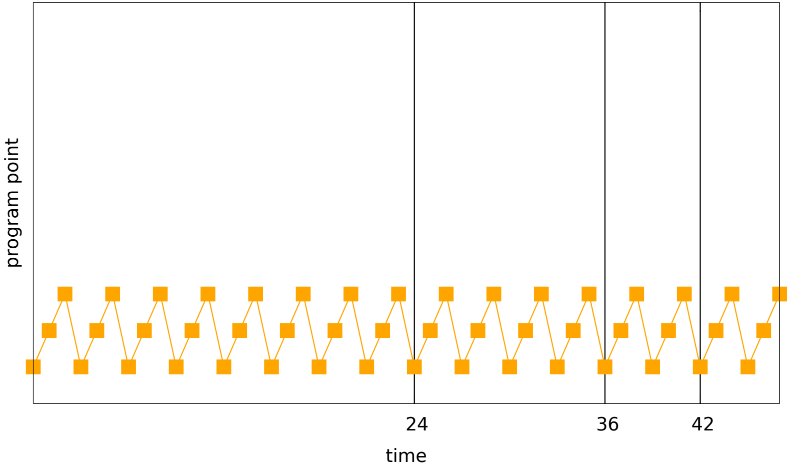

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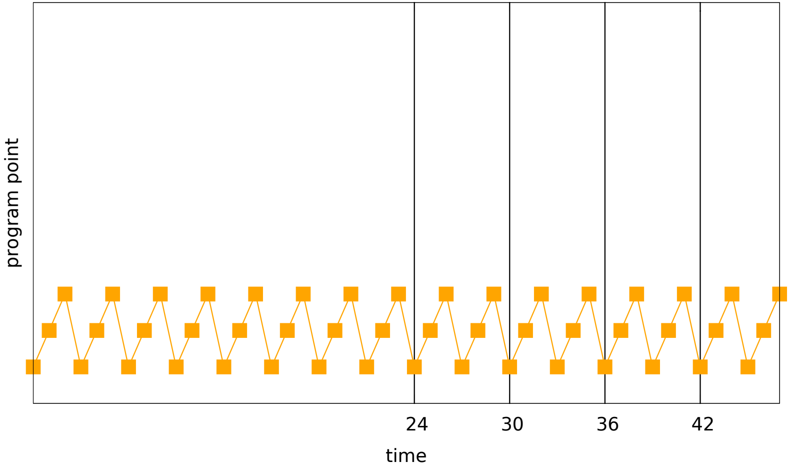

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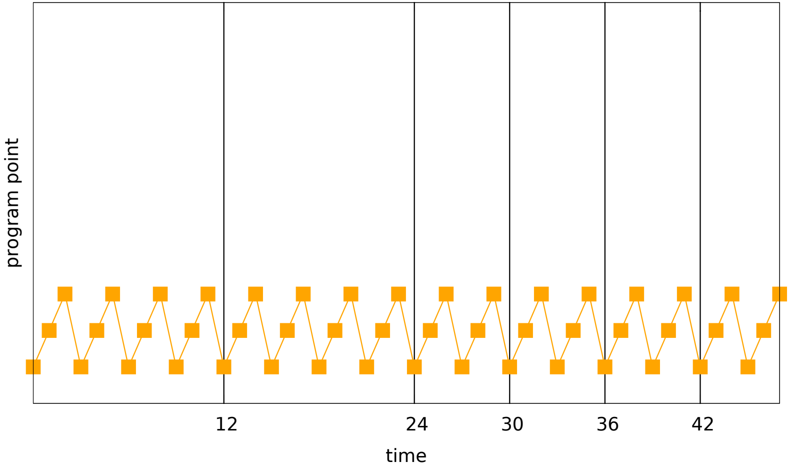

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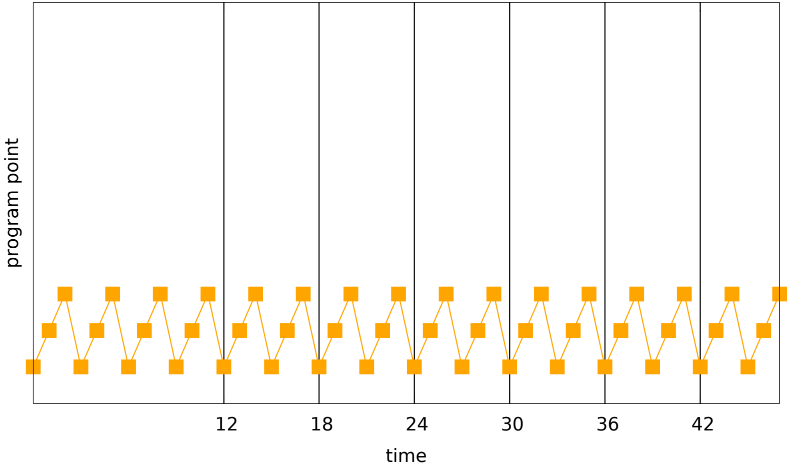

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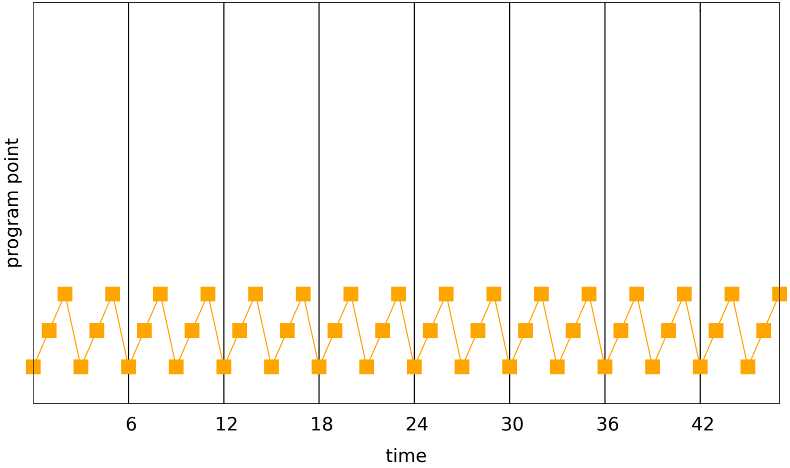
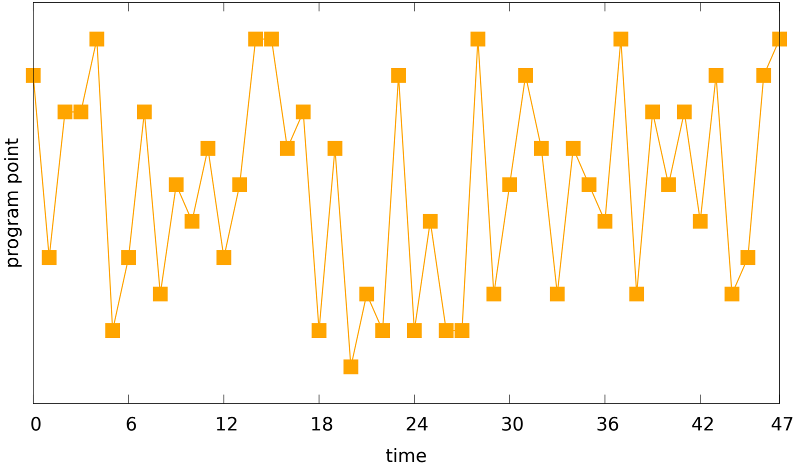


Execution Trace of Arbitrary Code



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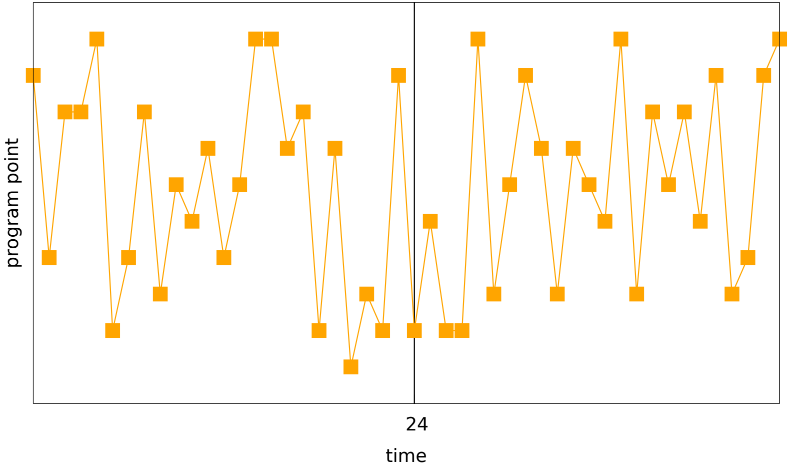

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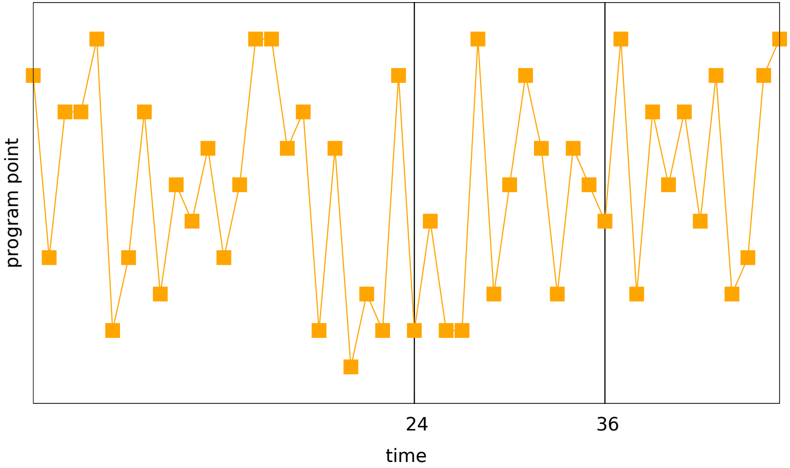

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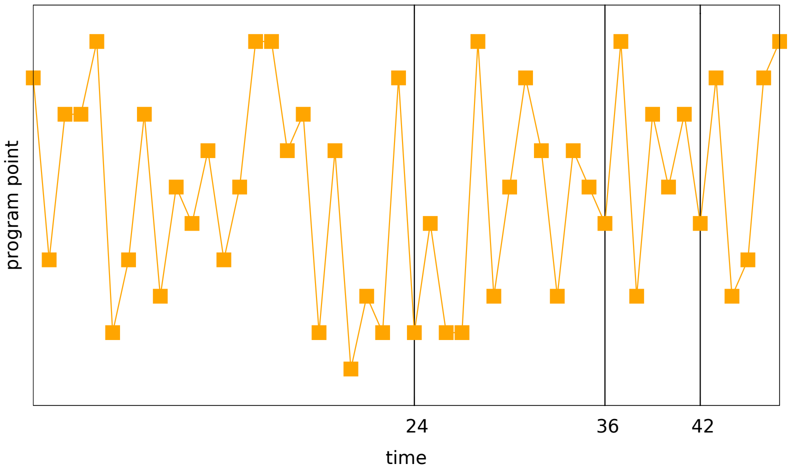

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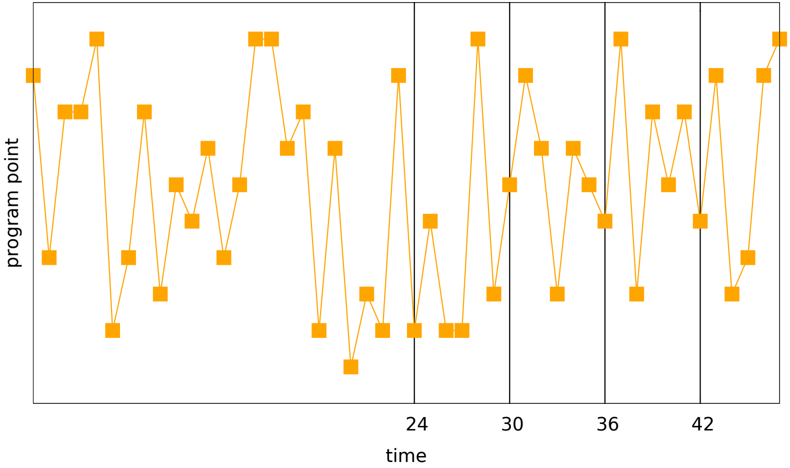

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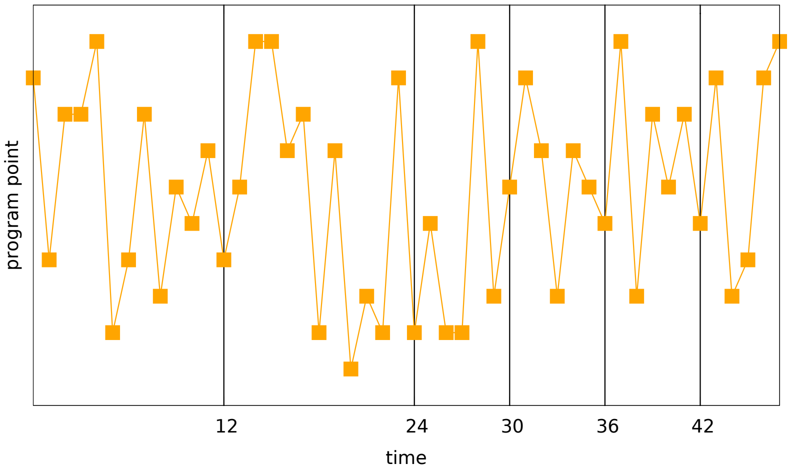

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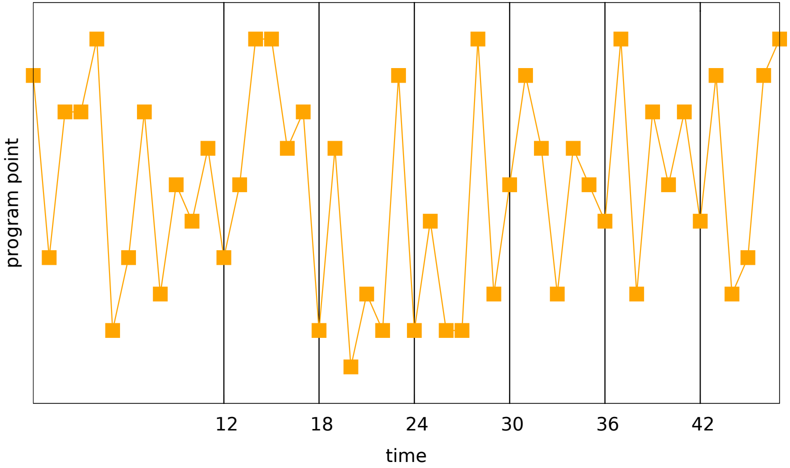

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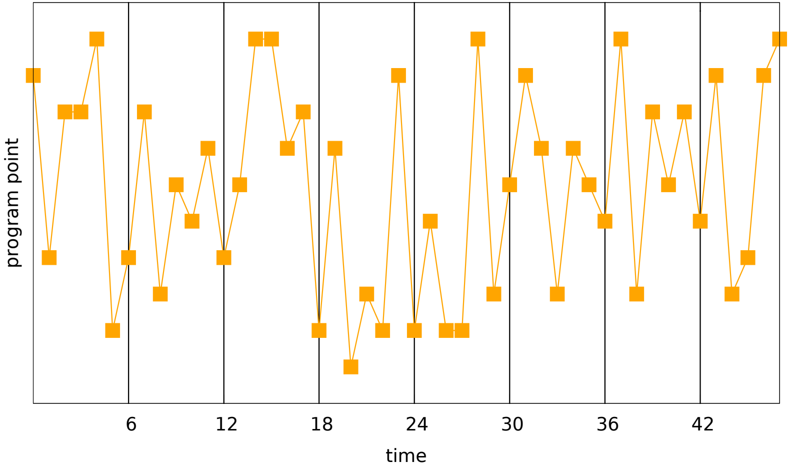

Key Challenges

Need to interleave generation of the network with forward and backward passes through the network.

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Portions of the network need to be (re)generated, and (re)evaluated with forward and backward passes, multiple times and out of order.

Key Idea

```
function main(w)
   local x = f(w)
   local y = h(g(x))
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$\rightsquigarrow$

```
function main(w)
    for i = 1, 5
    if i==1 then
        local x = f(w)
    elseif i==2 then
        local t = g(x)
    elseif i==3 then
        local y = h(t)
    elseif i==4 then
        local z = p(y)
    elseif i==5 then
        return z
    end
end
```

Core Language

$$e ::= c \mid x \mid \lambda x.e \mid e_1\ e_2 \mid \textbf{if}\ e_1\ \textbf{then}\ e_2\ \textbf{else}\ e_3 \mid \diamond e \mid e_1 \bullet e_2$$

Adding AD Operators to the Core Language

$$\overleftarrow{\mathcal{J}} : f\ x\ \grave{y} \mapsto (y, \grave{x})$$

$$\checkmark\!\!\!\mathcal{J} : f\ x\ \grave{y} \mapsto (y, \grave{x})$$

Algorithm for Divide-and-Conquer Checkpointing

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast): $(y, \grave{x}) = \overleftarrow{\mathcal{J}}\ f\ x\ \grave{y}$ (step 0)

inductive case: $h \circ g = f$ (step 1)

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3. save the state of the interrupted computation as a capsule
4. resume an interrupted computation from a capsule

General-Purpose Interruption and Resumption Interface

PRIMOPS $f\ x \mapsto l$	Return the number l of evaluation steps needed to compute $y = f(x)$.
INTERRUPT $f\ x\ l \mapsto z$	Run the first l steps of the computation of $f(x)$ and return a capsule z.
RESUME $z \mapsto y$	If $z = (\text{INTERRUPT}\ f\ x\ l)$, return $y = f(x)$.

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$\text{INTERRUPT}\, f\ x\ l \mapsto z$	Run the first l steps of the computation of $f(x)$ and return a capsule z.
$\text{RESUME}\ z \mapsto y$	If $z = (\text{INTERRUPT}\, f\ x\ l)$, return $y = f(x)$.

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Algorithm for Divide-and-Conquer Checkpointing

via General-Purpose Interruption and Resumption Interface

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast): $(y, \grave{x}) = \overleftarrow{\mathcal{J}}\ f\ x\ \grave{y}$ (step 0)

inductive case: $h \circ g = f$ (step 1)

$z = g\ x$ (step 2)

$(y, \grave{z}) = \overset{\checkmark}{\mathcal{J}}\ h\ z\ \grave{y}$ (step 3)

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inductive case:	$l = \text{PRIMOPS}\; f\; x$	(step 1)
	$z = \text{INTERRUPT}\; f\; x\; \lfloor \frac{l}{2} \rfloor$	(step 2)
	$(y, \grave{z}) = \overset{\checkmark}{\mathcal{J}}\; h\; z\; \grave{y}$	(step 3)
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	$(y, \grave{z}) = \overset{\checkmark}{\mathcal{J}}\ (\lambda z.\text{RESUME}\ z)\ z\ \grave{y}$	(step 3)
	$(z, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ (\lambda x.\text{INTERRUPT}\ f\ x\ \lfloor \frac{l}{2} \rfloor)\ x\ \grave{z}$	(step 4)

Example of CPS Conversion

```
function f(x)
   return q(p(g(x), h(x)))
end
```

$\rightsquigarrow$

```
function f(c, x)
   return g(function(t1)
               return h(function(t2)
                           return p(function(t3)
                                       return q(c, t3)
                                    end, t1, t2)
                        end, x)
            end, x)
end
```

Implementation

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1. Convert source program to CPS.

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CPS Conversion as a Program Transformation

$$\ulcorner x | k \urcorner \rightsquigarrow k\ x$$

$$\ulcorner (\lambda x.e) | k \urcorner \rightsquigarrow k\ (\lambda k'\ x. \ulcorner e | k' \urcorner)$$

$$\ulcorner (e_1\ e_2) | k \urcorner \rightsquigarrow \ulcorner e_1 | (\lambda x_1. \ulcorner e_2 | (\lambda x_2.(x_1\ k\ x_2)) \urcorner) \urcorner$$

$$e_0 \rightsquigarrow \ulcorner e_0 | (\lambda x.x) \urcorner$$

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CPS Conversion as a Program Transformation

$$\ulcorner x|{\color{red}k}\urcorner \rightsquigarrow k\ x$$
$$\ulcorner(\lambda x.e)|k\urcorner \rightsquigarrow k\ (\lambda k'\ x.\ulcorner e|k'\urcorner)$$
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\ulcorner (e_1\ e_2)|k \urcorner &\rightsquigarrow \ulcorner e_1|(\lambda x_1.\ulcorner e_2|(\lambda x_2.(x_1\ k\ x_2))\urcorner)\urcorner \\
e_0 &\rightsquigarrow \ulcorner e_0|(\lambda x.x)\urcorner
\end{aligned}$$

CPS Conversion as a Program Transformation

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\ulcorner x|k \urcorner &\leadsto k\ x \\
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\ulcorner ({\color{red}e_1}\ e_2)|k \urcorner &\leadsto \ulcorner e_1|(\lambda {\color{red}x_1}.\ulcorner e_2|(\lambda x_2.({\color{red}x_1}\ k\ x_2))\urcorner)\urcorner\\
e_0 &\leadsto \ulcorner e_0|(\lambda x.x)\urcorner
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\end{aligned}$$

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4. Combine with general-purpose interruption and resumption interface and $\sqrt{\mathcal{J}}$ written in C.
5. Compile to machine code.

Treading Step Counts and Limits in CPS Conversion

$$
\begin{aligned}
\ulcorner x|k \quad \urcorner &\leadsto k \quad x\\
\ulcorner(\lambda x.e)|k \quad \urcorner &\leadsto k \quad (\lambda k \quad x.\ulcorner e|k \quad \urcorner)\\
\ulcorner(e_1\ e_2)|k \quad \urcorner &\leadsto \ulcorner e_1|(\lambda \quad x_1.\\
&\qquad \ulcorner e_2|(\lambda \quad x_2.\\
&\qquad\qquad (x_1\ k \quad x_2)),\\
&\qquad\qquad \urcorner),\\
&\qquad\qquad \urcorner
\end{aligned}
$$

Treading Step Counts and Limits in CPS Conversion

$$\ulcorner x | k \urcorner \rightsquigarrow k\ x$$

$$\ulcorner (\lambda x.e) | k \urcorner \rightsquigarrow k\ (\lambda k\ x.\ulcorner e | k \urcorner)$$

$$\ulcorner (e_1\ e_2) | k \urcorner \rightsquigarrow \ulcorner e_1 | (\lambda\ x_1.\ \ulcorner e_2 | (\lambda\ x_2.\ (x_1\ k\ x_2)), \urcorner), \urcorner$$

Treading Step Counts and Limits in CPS Conversion

$$
\begin{aligned}
\ulcorner x | k, n \ \lrcorner &\rightsquigarrow k\ (n+1)\ \ x \\
\ulcorner (\lambda x.e) | k, n \ \lrcorner &\rightsquigarrow k\ (n+1)\ \ (\lambda k\ n\ \ x.\ulcorner e | k, n\ \lrcorner) \\
\ulcorner (e_1\ e_2) | k, n \ \lrcorner &\rightsquigarrow \ulcorner e_1 | (\lambda n\ \ x_1. \\
&\qquad \ulcorner e_2 | (\lambda n\ \ x_2. \\
&\qquad\qquad (x_1\ k\ n\ \ x_2)), \\
&\qquad n\ \lrcorner), \\
&\quad (n+1)\ \lrcorner
\end{aligned}
$$

Treading Step Counts and Limits in CPS Conversion

$$
\begin{aligned}
\ulcorner x|k,n,\color{red}{l}\urcorner &\rightsquigarrow\ k\ (n+1)\ \color{red}{l}\ x\\
\ulcorner(\lambda x.e)|k,n,\color{red}{l}\urcorner &\rightsquigarrow\ k\ (n+1)\ \color{red}{l}\ (\lambda k\ n\ \color{red}{l}\ x.\ulcorner e|k,n,\color{red}{l}\urcorner)\\
\ulcorner(e_1\ e_2)|k,n,\color{red}{l}\urcorner &\rightsquigarrow\ \ulcorner e_1|(\lambda n\ \color{red}{l}\ x_1.\\
&\qquad\qquad \ulcorner e_2|(\lambda n\ \color{red}{l}\ x_2.\\
&\qquad\qquad\qquad (x_1\ k\ n\ \color{red}{l}\ x_2)),\\
&\qquad\qquad\quad n,\color{red}{l}\urcorner),\\
&\qquad\qquad (n+1),\color{red}{l}\urcorner
\end{aligned}
$$

Treading Step Counts and Limits in CPS Conversion

$$
\begin{aligned}
\ulcorner x|k,n,l\urcorner &\leadsto k\;(n+1)\;l\;x\\
\ulcorner(\lambda x.e)|k,n,l\urcorner &\leadsto k\;(n+1)\;l\;(\lambda k\;n\;l\;x.\ulcorner e|k,n,l\urcorner)\\
\ulcorner(e_1\;e_2)|k,n,l\urcorner &\leadsto \ulcorner e_1|(\lambda n\;l\;x_1.\\
&\qquad\quad \ulcorner e_2|(\lambda n\;l\;x_2.\\
&\qquad\qquad\quad (x_1\;k\;n\;l\;x_2)),\\
&\qquad\qquad n,l\urcorner),\\
&\qquad (n+1),l\urcorner
\end{aligned}
$$

$$\langle\!\langle e\rangle\!\rangle_{k,n,l} \leadsto \textbf{if}\; n=l\;\textbf{then}\;[\![k,\lambda k\;n\;l\;\_.e]\!]\;\textbf{else}\;e$$

Treading Step Counts and Limits in CPS Conversion

$$
\begin{aligned}
\ulcorner x|k,n,l\urcorner &\leadsto \langle\!\langle k\ (n+1)\ l\ x\rangle\!\rangle_{k,n,l}\\
\ulcorner(\lambda x.e)|k,n,l\urcorner &\leadsto \langle\!\langle k\ (n+1)\ l\ (\lambda k\ n\ l\ x.\ulcorner e|k,n,l\urcorner)\rangle\!\rangle_{k,n,l}\\
\ulcorner(e_1\ e_2)|k,n,l\urcorner &\leadsto \langle\!\langle \ulcorner e_1|(\lambda n\ l\ x_1.\\
&\qquad\qquad \ulcorner e_2|(\lambda n\ l\ x_2.\\
&\qquad\qquad\qquad (x_1\ k\ n\ l\ x_2)),\\
&\qquad\qquad n,l\urcorner),\\
&\qquad (n+1),l\urcorner\rangle\!\rangle_{k,n,l}
\end{aligned}
$$

$$\langle\!\langle e\rangle\!\rangle_{k,n,l} \leadsto \textbf{if}\ n=l\ \textbf{then}\ [\![k,\lambda k\ n\ l\ \_.e]\!]\ \textbf{else}\ e$$

Treading Step Counts and Limits in CPS Conversion

$$\ulcorner x|k,n,l\urcorner \rightsquigarrow \langle\!\langle k\ (n+1)\ l\ x\rangle\!\rangle_{k,n,l}$$

$$\ulcorner(\lambda x.e)|k,n,l\urcorner \rightsquigarrow \langle\!\langle k\ (n+1)\ l\ (\lambda k\ n\ l\ x.\ulcorner e|k,n,l\urcorner)\rangle\!\rangle_{k,n,l}$$

$$\begin{aligned}\ulcorner(e_1\ e_2)|k,n,l\urcorner \rightsquigarrow \langle\!\langle \ulcorner e_1|(\lambda n\ l\ x_1.\\ \ulcorner e_2|(\lambda n\ l\ x_2.\\ (x_1\ k\ n\ l\ x_2)),\\ n,l\urcorner),\\ (n+1),l\urcorner\rangle\!\rangle_{k,n,l}\end{aligned}$$

$$\vdots$$

$$\langle\!\langle e\rangle\!\rangle_{k,n,l} \rightsquigarrow \textbf{if}\ n=l\ \textbf{then}\ [\![k,\lambda k\ n\ l\ \_.e]\!]\ \textbf{else}\ e$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\begin{aligned}
\text{PRIMOPS}\, f\ x &= \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ f\ x\\
\text{INTERRUPT}\, f\ x\ l &= \mathcal{A}\ (\lambda n\ l\ v.v)\ 0\ l\, f\ x\\
\text{RESUME}\ [\![k, f]\!] &= \mathcal{A}\ k\ 0\ \infty\ f\ \bot
\end{aligned}$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\text{PRIMOPS}\ \textcolor{red}{f}\ x = \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ \textcolor{red}{f}\ x$$
$$\text{INTERRUPT}\ f\ x\ l = \mathcal{A}\ (\lambda n\ l\ v.v)\ 0\ l\ f\ x$$
$$\text{RESUME}\ [\![k, f]\!] = \mathcal{A}\ k\ 0\ \infty\ f\ \bot$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\text{PRIMOPS}\, f\ \color{red}{x} = \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ f\ \color{red}{x}$$

$$\text{INTERRUPT}\, f\ x\ l = \mathcal{A}\ (\lambda n\ l\ v.v)\ 0\ l\ f\ x$$

$$\text{RESUME}\ [\![k, f]\!] = \mathcal{A}\ k\ 0\ \infty\ f\ \bot$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\text{PRIMOPS}\, f\ x = \mathcal{A}\ (\lambda n\ l\ v.n)\ {\color{red}0}\ \infty\ f\ x$$
$$\text{INTERRUPT}\, f\ x\ l = \mathcal{A}\ (\lambda n\ l\ v.v)\ 0\ l\ f\ x$$
$$\text{RESUME}\ [\![k,f]\!] = \mathcal{A}\ k\ 0\ \infty\ f\ \bot$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\text{PRIMOPS}\, f\; x = \mathcal{A}\; (\lambda n\; l\; v.n)\; 0\; \infty\; f\; x$$

$$\text{INTERRUPT}\, f\; x\; l = \mathcal{A}\; (\lambda n\; l\; v.v)\; 0\; l\; f\; x$$

$$\text{RESUME}\; [\![k, f]\!] = \mathcal{A}\; k\; 0\; \infty\; f\; \bot$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\text{PRIMOPS}\, f\ x = \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ f\ x$$
$$\text{INTERRUPT}\, f\ x\ l = \mathcal{A}\ (\lambda n\ l\ v.v)\ 0\ l\ f\ x$$
$$\text{RESUME}\ [\![k, f]\!] = \mathcal{A}\ k\ 0\ \infty\ f\ \bot$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\begin{aligned}
\text{PRIMOPS}\; f\; x &= \mathcal{A}\;(\lambda n\; l\; v.n)\; 0\; \infty\; f\; x\\
\text{INTERRUPT}\; f\; x\; l &= \mathcal{A}\;(\lambda n\; l\; v.v)\; 0\; l\; f\; x\\
\text{RESUME}\; [\![k, f]\!] &= \mathcal{A}\; k\; 0\; \infty\; f\; \bot
\end{aligned}$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\textsc{primops}\ f\ x = \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ f\ x$$

$$\textsc{interrupt}\ f\ {\color{red}x}\ l = \mathcal{A}\ (\lambda n\ l\ v.v)\ 0\ l\ f\ {\color{red}x}$$

$$\textsc{resume}\ [\![k, f]\!] = \mathcal{A}\ k\ 0\ \infty\ f\ \bot$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$
\begin{aligned}
\text{PRIMOPS}\, f\ x &= \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ f\ x\\
\text{INTERRUPT}\, f\ x\ l &= \mathcal{A}\ (\lambda n\ l\ v.v)\ \color{red}{0}\ l\, f\ x\\
\text{RESUME}\ [\![k, f]\!] &= \mathcal{A}\ k\ 0\ \infty\ f\ \bot
\end{aligned}
$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\text{PRIMOPS}\, f\ x = \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ f\ x$$
$$\text{INTERRUPT}\, f\ x\ {\color{red}l} = \mathcal{A}\ (\lambda n\ l\ v.v)\ 0\ {\color{red}l}\ f\ x$$
$$\text{RESUME}\ [\![k,f]\!] = \mathcal{A}\ k\ 0\ \infty\ f\ \bot$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\begin{aligned}
\text{PRIMOPS}\, f\ x &= \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ f\ x\\
\text{INTERRUPT}\, f\ x\ l &= \mathcal{A}\ (\lambda n\ l\ v.v)\ 0\ l\ f\ x\\
\text{RESUME}\ [\![k,f]\!] &= \mathcal{A}\ k\ 0\ \infty\ f\ \bot
\end{aligned}$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\text{PRIMOPS}\, f\ x = \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ f\ x$$
$$\text{INTERRUPT}\, f\ x\ l = \mathcal{A}\ (\lambda n\ l\ v.v)\ 0\ l\ f\ x$$
$$\text{RESUME}\ [\![k, f]\!] = \mathcal{A}\ k\ 0\ \infty\ f\ \bot$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\text{PRIMOPS}\, f\ x = \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ f\ x$$
$$\text{INTERRUPT}\, f\ x\ l = \mathcal{A}\ (\lambda n\ l\ v.v)\ 0\ l\ f\ x$$
$$\text{RESUME}\ [\![k, f]\!] = \mathcal{A}\ k\ 0\ \infty\ f\ \bot$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\text{PRIMOPS}\,f\;x = \mathcal{A}\;(\lambda n\;l\;v.n)\;0\;\infty\;f\;x$$
$$\text{INTERRUPT}\,f\;x\;l = \mathcal{A}\;(\lambda n\;l\;v.v)\;0\;l\,f\;x$$
$$\text{RESUME}\;[\![k,f]\!] = \mathcal{A}\;k\;0\;\infty\;f\;\bot$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\text{PRIMOPS}\, f\ x = \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ f\ x$$
$$\text{INTERRUPT}\, f\ x\ l = \mathcal{A}\ (\lambda n\ l\ v.v)\ 0\ l\ f\ x$$
$$\text{RESUME}\ [\![k, f]\!] = \mathcal{A}\ k\ {\color{red}0}\ \infty\ f\ \bot$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\text{PRIMOPS}\, f\; x = \mathcal{A}\; (\lambda n\; l\; v.n)\; 0\; \infty\; f\; x$$
$$\text{INTERRUPT}\, f\; x\; l = \mathcal{A}\; (\lambda n\; l\; v.v)\; 0\; l\; f\; x$$
$$\text{RESUME}\, [\![k, f]\!] = \mathcal{A}\; k\; 0\; \infty\; f\; \bot$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$
\begin{aligned}
\text{PRIMOPS}\, f\ x &= \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ f\ x\\
\text{INTERRUPT}\, f\ x\ l &= \mathcal{A}\ {\color{red}(\lambda n\ l\ v.v)}\ 0\ l\, f\ x\\
\text{RESUME}\ [\![k, f]\!] &= \mathcal{A}\ {\color{red}k}\ 0\ \infty\ f\ \bot
\end{aligned}
$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$\begin{aligned}
\text{PRIMOPS}\, f\ x &= \mathcal{A}\ (\lambda n\ l\ v.n)\ 0\ \infty\ f\ x\\
\text{INTERRUPT}\, f\ x\ l &= \mathcal{A}\ (\lambda n\ l\ {\color{red}v}.{\color{red}v})\ 0\ l\, f\ x\\
\text{RESUME}\ [\![k, f]\!] &= \mathcal{A}\ k\ 0\ \infty\ f\ \bot
\end{aligned}$$

Implementation

1. Convert source program to CPS.
2. Thread step count and limit.
3. Translate CPS to C.
4. Combine with general-purpose interruption and resumption interface and $\sqrt{\mathcal{J}}$ written in C.
5. Compile to machine code.

Code Generation

$$
\begin{aligned}
\mathcal{S}\,\pi\,() &= \texttt{null\_constant}\\
\mathcal{S}\,\pi\,\textbf{true} &= \texttt{true\_constant}\\
\mathcal{S}\,\pi\,\textbf{false} &= \texttt{false\_constant}\\
\mathcal{S}\,\pi\,(c_1,c_2) &= \texttt{cons}\,\texttt{((}\mathcal{S}\,\pi\,c_1\texttt{),\ \ (}\mathcal{S}\,\pi\,c_1\texttt{))}\\
\mathcal{S}\,\pi\,n & \\
\mathcal{S}\,\pi\,\texttt{`k'} &= \texttt{continuation}\\
\mathcal{S}\,\pi\,\texttt{`n'} &= \texttt{count}\\
\mathcal{S}\,\pi\,\texttt{`l'} &= \texttt{limit}\\
\mathcal{S}\,\pi\,\texttt{`x'} &= \texttt{argument}\\
\mathcal{S}\,\pi\,x &= \texttt{as\_closure(target)->environment[}\pi\,x\texttt{]}
\end{aligned}
$$

Code Generation

$\mathcal{S}\,\pi\,(\lambda_3 n\, l\, x.e)$ = ({

```
    thing function(thing target,
                   thing count,
                   thing limit,
                   thing argument) {
    return (S (φ e) e);
    }
    thing lambda = (thing)GC_malloc(sizeof(struct {
      enum tag tag;
      struct {
        thing (*function)();
        unsigned n;
        thing environment[|φ e|];
      }))
    }
    set_closure(lambda);
    as_closure(lambda)->function = &function;
    as_closure(lambda)->n = |φ e|;
    as_closure(lambda)->environment[0] = S π (φ e)_0
    ⋮
    as_closure(lambda)->environment[|φ e| − 1] = S π (φ e)_{|φ e|−1}
    lambda;
  })
```

Code Generation

$\mathcal{S}\,\pi\,(\lambda_4 k\,n\,l\,x.e)$ = ({

```
              thing function(thing target,
                             thing continuation,
                             thing count,
                             thing limit,
                             thing argument) {
              return ($\mathcal{S}\,(\phi\,e)\,e$);
              }
              thing lambda = (thing)GC_malloc(sizeof(struct {
                enum tag tag;
                struct {
                  thing (*function)();
                  unsigned n;
                  thing environment[$|\phi\,e|$];
                }))
              }
              set_closure(lambda);
              as_closure(lambda)->function = &function;
              as_closure(lambda)->n = $|\phi\,e|$;
              as_closure(lambda)->environment[0] = $\mathcal{S}\,\pi\,(\phi\,e)_0$
              ⋮
              as_closure(lambda)->environment[$|\phi\,e|-1$] = $\mathcal{S}\,\pi\,(\phi\,e)_{|\phi\,e|-1}$
              lambda;
            })
```

Code Generation

$$\mathcal{S}\,\pi\,(e_1\;e_2\;e_3\;e_4) = \texttt{continuation\_apply}\texttt{(}(\mathcal{S}\,\pi\,e_1)\texttt{,}\;(\mathcal{S}\,\pi\,e_2)\texttt{,}\;(\mathcal{S}\,\pi\,e_3)\texttt{,}\;(\mathcal{S}\,\pi\,e_4)\texttt{)}$$

$$\mathcal{S}\,\pi\,(e_1\;e_2\;e_3\;e_4\;e_5) = \texttt{converted\_apply}\texttt{(}(\mathcal{S}\,\pi\,e_1)\texttt{,}\;(\mathcal{S}\,\pi\,e_2)\texttt{,}\;(\mathcal{S}\,\pi\,e_3)\texttt{,}\;(\mathcal{S}\,\pi\,e_4)\texttt{,}\;(\mathcal{S}\,\pi\,e_5)\texttt{)}$$

$$\mathcal{S}\,\pi\,(\textbf{if}\;e_1\;\textbf{then}\;e_2\;\textbf{else}\;e_3) = \texttt{(!is\_false(}(\mathcal{S}\,\pi\,e_1)\texttt{)?}(\mathcal{S}\,\pi\,e_2)\texttt{:}(\mathcal{S}\,\pi\,e_3)\texttt{)}$$

$$\mathcal{S}\,\pi\,(\diamond e) = (\mathcal{N}\,\diamond)\texttt{(}(\mathcal{S}\,\pi\,e)\texttt{)}$$

$$\mathcal{S}\,\pi\,(e_1 \bullet e_2) = (\mathcal{N}\,\bullet)\texttt{(}(\mathcal{S}\,\pi\,e_1)\texttt{,}\;(\mathcal{S}\,\pi\,e_2)\texttt{)}$$

$$\mathcal{S}\,\pi\,(\overrightarrow{\mathcal{J}}\;e_1\;e_2\;e_3) = (\mathcal{N}\,\overrightarrow{\mathcal{J}})\texttt{(}(\mathcal{S}\,\pi\,e_1)\texttt{,}\;(\mathcal{S}\,\pi\,e_2)\texttt{,}\;(\mathcal{S}\,\pi\,e_3)\texttt{)}$$

$$\mathcal{S}\,\pi\,(\overleftarrow{\mathcal{J}}\;e_1\;e_2\;e_3) = (\mathcal{N}\,\overleftarrow{\mathcal{J}})\texttt{(}(\mathcal{S}\,\pi\,e_1)\texttt{,}\;(\mathcal{S}\,\pi\,e_2)\texttt{,}\;(\mathcal{S}\,\pi\,e_3)\texttt{)}$$

$$\mathcal{S}\,\pi\,(\stackrel{\checkmark}{\mathcal{J}}\;e_1\;e_2\;e_3) = (\mathcal{N}\,\stackrel{\checkmark}{\mathcal{J}})\texttt{(}(\mathcal{S}\,\pi\,e_1)\texttt{,}\;(\mathcal{S}\,\pi\,e_2)\texttt{,}\;(\mathcal{S}\,\pi\,e_3)\texttt{)}$$

Implementation

1. Convert source program to CPS.
2. Thread step count and limit.
3. Translate CPS to C.
4. Combine with general-purpose interruption and resumption interface and $\checkmark{}$ $\mathcal{J}$ written in C.
5. Compile to machine code.

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

$$
\begin{aligned}
\text{PRIMOPS}\, f\; x &= \mathcal{A}\; (\lambda n\; l\; v.n)\; 0\; \infty\; f\; x\\
\text{INTERRUPT}\, f\; x\; l &= \mathcal{A}\; (\lambda n\; l\; v.v)\; 0\; l\; f\; x\\
\text{RESUME}\, [\![k,f]\!] &= \mathcal{A}\; k\; 0\; \infty\; f\; \bot
\end{aligned}
$$

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

written in C

```
static thing lambda_expression_that_returns_x
(thing f, thing n, thing l, thing x) {
  return x;
}

static thing lambda_expression_that_returns_n
(thing f, thing n, thing l, thing x) {
  return n;
}

static thing lambda_expression_that_resumes
(thing f, thing continuation, thing n, thing l, thing x) {
  if (!is_interrupt(x)) internal_error();
  return converted_apply(as_interrupt(x)->closure,
                         as_interrupt(x)->continuation,
                         make_real(0.0),
                         l,
                         null_constant);
}
```

Implementation of the General-Purpose Interruption and Resumption Interface with Threaded CPS

```
static unsigned long primops(thing f, thing x) {
  thing result = converted_apply(f,
                                 continuation_that_returns_n,
                                 make_real(0.0),
                                 make_real(HUGE_VAL),
                                 x);
  else if (is_real(result)) return (unsigned long)as_real(result);
}

static thing interrupt(thing f, thing x, thing l) {
  thing result = converted_apply(f,
                                 continuation_that_returns_x,
                                 make_real(0.0),
                                 l,
                                 x);
  if (!is_interrupt(result)) internal_error();
  return result;
}
```

Algorithm for Divide-and-Conquer Checkpointing

To compute $(y, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ f\ x\ \grave{y}$:

base case ($f\ x$ fast):	$(y, \grave{x}) = \overleftarrow{\mathcal{J}}\ f\ x\ \grave{y}$	(step 0)
inductive case:	$h \circ g = f$	(step 1)
	$z = g\ x$	(step 2)
	$(y, \grave{z}) = \overset{\checkmark}{\mathcal{J}}\ h\ z\ \grave{y}$	(step 3)
	$(z, \grave{x}) = \overset{\checkmark}{\mathcal{J}}\ g\ x\ \grave{z}$	(step 4)

Algorithm for Divide-and-Conquer Checkpointing

written in C

```
static thing checkpoint_starj(thing f, thing x, thing y_cotangent)
{
  thing loop(thing f, thing x, thing y_cotangent, unsigned long l) {
    if (l<=base_case_duration) return ternary_starj(f, x, y_cotangent);
    else {
      thing u = interrupt(f, x, make_real(l/2));
      thing y_u_cotangent = loop(closure_that_resumes, u, y_cotangent, l-l/2);
      if (!is_pair(y_u_cotangent)) internal_error();
      thing u_x_cotangent =
        loop(make_closure_for_interrupt(f, l/2),
             x,
             as_pair(y_u_cotangent)->cdr,
             l/2);
      if (!is_pair(u_x_cotangent)) internal_error();
      return cons(as_pair(y_u_cotangent)->car,
                  as_pair(u_x_cotangent)->cdr);
    }
  }
  return loop(f, x, y_cotangent, primops(f, x));
}
```

Implementation

1. Convert source program to CPS.
2. Thread step count and limit.
3. Translate CPS to C.
4. Combine with general-purpose interruption and resumption interface and $\sqrt{\mathcal{J}}$ written in C.
5. Compile to machine code.

Determinant Example

```
(define (car (cons car cdr)) car)

(define (cdr (cons car cdr)) cdr)

(define (matrix-rows a)
 (if (null? a) 0 (+ (matrix-rows (cdr a)) 1)))

(define (list-ref l i)
 (if (zero? i) (car l) (list-ref (cdr l) (- i 1))))

(define (matrix-ref a i j) (list-ref (list-ref a i) j))

(define (list-set l i x)
 (if (zero? i)
     (cons x (cdr l))
     (cons (car l) (list-set (cdr l) (- i 1) x))))

(define (matrix-set a i j x)
 (list-set a i (list-set (list-ref a i) j x)))

(define (map-n f n)
 (if (zero? n) '() (cons (f (- n 1)) (map-n f (- n 1)))))

(define (identity-matrix n)
 (map-n (lambda (i) (map-n (lambda (j) (if (= i j) 1 0)) n)) n))
```

Determinant Example

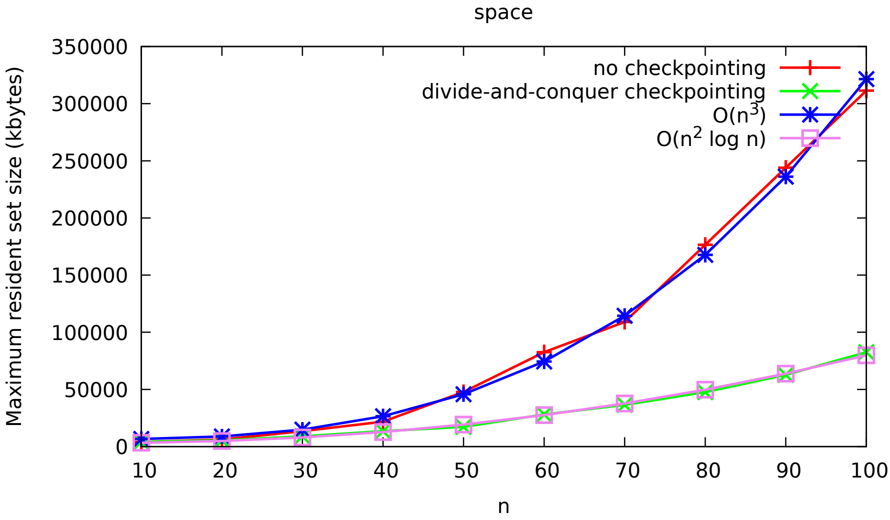
```
(define (determinant a)
 (let ((n (matrix-rows a)))
  (let loop ((i 0) (b a) (d 1))
   (if (= i n)
       d
       (let* ((c (matrix-ref b i i))
              (b (let loop ((j i) (b b))
                  (if (= j n)
                      b
                      (loop (+ j 1) (matrix-set b i j (/ (matrix-ref b i j) c)))))))
        (loop (+ i 1)
              (let loop ((j (+ i 1)) (b b))
               (if (= j n)
                   b
                   (loop (+ j 1)
                         (let ((e (matrix-ref b j i)))
                          (let loop ((k (+ i 1)) (b b))
                           (if (= k n)
                               b
                               (loop (+ k 1)
                                     (matrix-set b j k
                                                 (- (matrix-ref b j k)
                                                    (* e (matrix-ref b i k)))))))))))
              (* d c)))))))

(write-real (determinant (cdr (checkpoint-*j determinant (identity-matrix (read-real)) 1))))
```

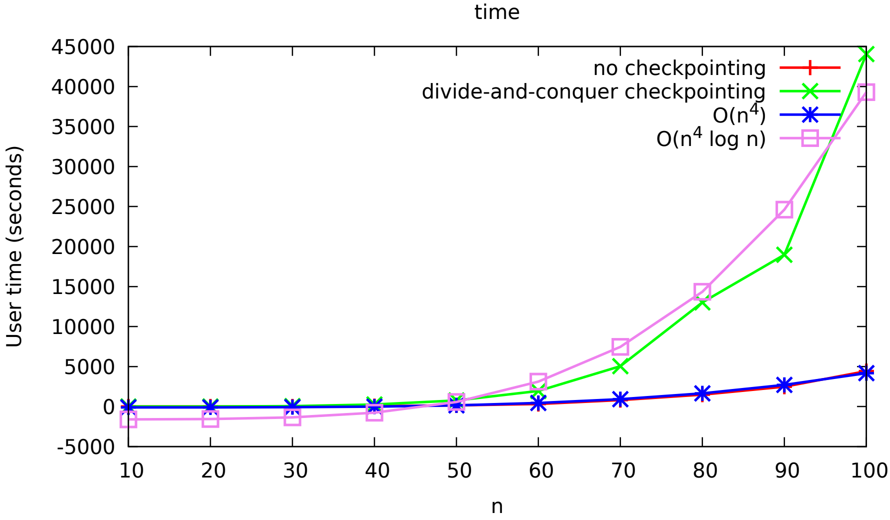
Complexity Analysis

	space	time
primal	$O(n^2)$	$O(n^4)$
no checkpointing	$O(n^3)$	$O(n^4)$
divide-and-conquer checkpointing	$O(n^2 \log n)$	$O(n^4 \log n)$

Space Usage of the Determinant Example



Time Usage of the Determinant Example



Three Reference Implementations

Three Reference Implementations

1. Interpreter using CPS evaluator

Three Reference Implementations

1. Interpreter using CPS evaluator
2. Hybrid compiler/interpreter using CPS conversion followed by direct-style evaluator

Three Reference Implementations

1. Interpreter using CPS evaluator
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Will release when manuscript is accepted.

Take-Home Message

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metaphor: a CPU is an instruction-execution loop

Thank You