The tension between convenience and performance in automatic differentiation

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Joint work with Barak Pearlmutter

Forward Mode

$$f = f_1 \circ \cdots \circ f_n$$

Forward Mode

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\begin{aligned}
f &= f_1 \circ \cdots \circ f_n \\
\mathcal{J}(f)(x_0) &= \mathcal{J}(f_n)(x_{n-1}) \times \cdots \times \mathcal{J}(f_1)(x_0)
\end{aligned}
$$

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$$\acute{x}_n = \mathcal{J}(f)(x_0) \times \acute{x}_0$$

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\begin{aligned}
x_1 &= f_1(x_0) \\
\acute{x}_1 &= \mathcal{J}(f_1)(x_0) \times \acute{x}_0 \\
&\vdots \\
x_n &= f_n(x_{n-1}) \\
\acute{x}_n &= \mathcal{J}(f_n)(x_{n-1}) \times \acute{x}_{n-1}
\end{aligned}
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Reverse Mode

$$f = f_1 \circ \cdots \circ f_n$$

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f &= f_1 \circ \cdots \circ f_n \\
\mathcal{J}(f)(x_0)^\top &= \mathcal{J}(f_1)(x_0)^\top \times \cdots \times \mathcal{J}(f_n)(x_{n-1})^\top
\end{aligned}
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Reverse Mode

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$$\mathcal{J}(f)(x_0)^\top = \mathcal{J}(f_1)(x_0)^\top \times \cdots \times \mathcal{J}(f_n)(x_{n-1})^\top$$

$$\grave{x}_0 = \mathcal{J}(f)(x_0)^\top \times \grave{x}_n$$

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\grave{x}_0 &= \mathcal{J}(f)(x_0)^\top \times \grave{x}_n
\end{aligned}
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\begin{aligned}
x_1 &= f_1(x_0) \\
&\vdots \\
x_n &= f_n(x_{n-1}) \\
\grave{x}_{n-1} &= \mathcal{J}(f_n)(x_{n-1}) \times \grave{x}_n \\
&\vdots \\
\grave{x}_0 &= \mathcal{J}(f_1)(x_0) \times \grave{x}_1
\end{aligned}
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Forward Mode by Overloading

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\begin{aligned}
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Forward Mode by Overloading

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\acute{x}_n &= \mathcal{J}(f_n)(x_{n-1}) \times \acute{x}_{n-1}
\end{aligned}
$$

$$x_i = f_i(x_{i-1})$$

Forward Mode by Overloading

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\acute{x}_1 &= \mathcal{J}(f_1)(x_0) \times \acute{x}_0\\
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x_n &= f_n(x_{n-1})\\
\acute{x}_n &= \mathcal{J}(f_n)(x_{n-1}) \times \acute{x}_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
x_i &= f_i(x_{i-1})\\
\langle x_i, \acute{x}_i \rangle &= \langle f_i(x_{i-1}), \mathcal{J}(f_i)(x_{i-1}) \times \acute{x}_{i-1} \rangle
\end{aligned}
$$

Forward Mode by Overloading

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$$
\begin{aligned}
x_i &= f_i(x_{i-1})\\
\langle x_i, \acute{x}_i \rangle &= \langle f_i(x_{i-1}), \mathcal{J}(f_i)(x_{i-1}) \times \acute{x}_{i-1} \rangle\\
\overrightarrow{x_i} &= \overrightarrow{f_i}\,(\overrightarrow{x_{i-1}})
\end{aligned}
$$

Implementation of Forward Mode by Overloading—I

```
(define-structure dual-number primal tangent)

(set! original+ +)

(define (+ x y)
 (dual-number
  (original+ (primal x) (primal y))
  (original+ (tangent x) (tangent y))))

(define (derivative f x)
 (tangent (f (dual-number x 1))))
```

Implementation of Forward Mode by Overloading—II

```
(set! original+ +)

(define (+ x y)
 (if (dual-number? x)
     (dual-number
       (original+ (primal x) (primal y))
       (original+ (tangent x) (tangent y)))
     (original+ x y)))
```

Implementation of Forward Mode by Overloading—III

```
(set! original+ +)

(define (+ x y)
 (if (dual-number? x)
     (dual-number
       (+ (primal x) (primal y))
       (+ (tangent x) (tangent y)))
     (original+ x y)))

(define (derivative2 f x)
 (tangent
  (tangent
   (f (dual-number
       (dual-number x 1)
       (dual-number 1 0))))))
```

Implementation of Forward Mode by Overloading—IV

```
(define +0 +)
(define (+1 x y)
 (dual-number
  (+0 (primal x) (primal y))
  (+0 (tangent x) (tangent y))))
(define (+2 x y)
 (dual-number
  (+1 (primal x) (primal y))
  (+1 (tangent x) (tangent y))))
⋮
(f0 x)
(tangent (f1 (dual-number x 1)))
(tangent
 (tangent
  (f2 (dual-number
       (dual-number x 1) (dual-number 1 0)))))
```

Implementation of Forward Mode by Overloading—V

```
(define +0 +)

(define (+1 xp xt yp yt)
 (values
  (+0 xp yp)
  (+0 xt yt)))

(define (+2 xpp xpt xtp xtt ypp ypt ytp ytt)
 (let-values ((zpp zpt (+1 xpp xpt ypp ypt))
              (ztp ttt (+1 xtp xtt ytp xtt)))
  (values zpp zpt ztp ztt)))
⋮
```

Dynamic Overloading: SCMUTILS

```
(define-structure bundle primal tangent)
(define (primal p) (if (bundle? p) (bundle-primal p) p))
(define (tangent p) (if (bundle? p) (bundle-tangent p) 0))

(define +
 (let ((+ +))
  (lambda (x1 x2)
   (make-bundle (+ (primal x1) (primal x2))
                (+ (tangent x1) (tangent x2))))))

(define *
 (let ((+ +) (* *))
  (lambda (x1 x2)
   (make-bundle (* (primal x1) (primal x2))
                (+ (* (primal x1) (tangent x2))
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(define ((derivative f) x) (tangent (f (make-bundle x 1))))
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(define (f x) (* 2 (* x (* x x))))
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(derivative f)
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(derivative (derivative f))
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Convenient

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Convenient but slow

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                   (* (tangent x1) (primal x2)))))))

(define ((derivative f) x) (tangent (f (make-bundle x 1))))

(define (f x) (* 2 (* x (* x x))))

(derivative f)
(derivative (derivative f))
(derivative (lambda (x) ... (derivative (lambda (y) ...) ...) ...) ...)
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```
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(define (tangent p) (if (bundle? p) (bundle-tangent p) 0))

(define ((derivative f) x)
 (fluid-let ((+ (lambda (x1 x2)
                (make-bundle (+ (primal x1) (primal x2))
                             (+ (tangent x1) (tangent x2)))))
            (* (lambda (x1 x2)
                (make-bundle (* (primal x1) (primal x2))
                             (+ (* (primal x1) (tangent x2))
                                (* (tangent x1) (primal x2)))))))
  (tangent (f (make-bundle x 1)))))
```

```
(define (f x) (* 2 (* x (* x x))))

(derivative f)
(derivative (derivative f))
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                  (make-bundle (* (primal x1) (primal x2))
                               (+ (* (primal x1) (tangent x2))
                                  (* (tangent x1) (primal x2)))))))
  (tangent (f (make-bundle x 1)))))
```

```
(define (f x) (* 2 (* x (* x x))))

(derivative f)
(derivative (derivative f))
(derivative (lambda (x) ... (derivative (lambda (y) ...) ...) ...) ...)
```

Convenient but slow

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end

function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

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Fast

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```

Fast but inconvenient

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
end
```

```
AD_TOP = f
```

```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
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end
```

Fast but inconvenient

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function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

Fast but inconvenient

Preprocessor: ADIFOR and TAPENADE

```
function f(x)
double precision x, f
f = 2.0d0*x*x*x
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```
AD_TOP = f
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AD_TOP = gf
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```

Fast but inconvenient

Preprocessor: ADIFOR and TAPENADE

```
function f(x)                              AD_TOP = f
double precision x, f                      AD_IVARS = x
f = 2.0d0*x*x*x                            AD_DVARS = f
end

function gf(x, gx, gresult)                AD_TOP = gf
double precision x, gx, gf, gresult        AD_IVARS = x, gx
gf = 2.0d0*x*x*x                           AD_DVARS = gf, gresult
gresult = 6.0d0*x*x*gx
end

function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
double precision x, gx, gx, ggx, ggf, gresult, gresult, ggresult
ggf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
gresult = 6.0d0*x*x*gx
ggresult = 6.0d0*x*x*ggx+12.0d0*x*gx*gx
end
```

Fast but inconvenient

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double precision x, f                   AD_IVARS = x
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gf = 2.0d0*x*x*x                        AD_DVARS = gf, gresult
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gf = 2.0d0*x*x*x                           AD_DVARS = gf, gresult
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function ggf(x, gx, gx, ggx, gresult, ggresult, gresult)
double precision x, gx, gx, ggx, ggf, gresult, gresult, ggresult
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AD_TOP = f
AD_IVARS = x
AD_DVARS = f
```

```
function gf(x, gx, gresult)
double precision x, gx, gf, gresult
gf = 2.0d0*x*x*x
gresult = 6.0d0*x*x*gx
end
```

```
AD_TOP = gf
AD_IVARS = x, gx
AD_DVARS = gf, gresult
AD_PREFIX = h
```

```
function hgf(x, hx, gx, hgx, gresult, hgresult, hresult)
double precision x, hx, gx, hgx, hgf, hresult, gresult, hgresult
hgf = 2.0d0*x*x*x
hresult = 6.0d0*x*x*hx
gresult = 6.0d0*x*x*gx
hgresult = 6.0d0*x*x*hgx+12.0d0*x*gx*hx
end
```

Fast but inconvenient

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...
```

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...
```

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

Slow

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

Slow

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

Slow

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

Slow

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
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... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

Slow and inconvenient

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

Slow and inconvenient

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

Slow and inconvenient

Static Overloading: FADBAD++

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double f(double x) {return 2*x*x*x;}
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... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

Slow and inconvenient

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

Slow and inconvenient

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

Slow and inconvenient

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...
```

Slow and inconvenient

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...

template <typename T>
T f(T x) {return 2*x*x*x;}
T x;
```

Slow and inconvenient

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...

template <typename T>
T f(T x) {return 2*x*x*x;}
T x;
```

Slow and inconvenient

Static Overloading: FADBAD++

```
double f(double x) {return 2*x*x*x;}
double x;
... f(x) ...

F<double> f(F<double> x) {return 2*x*x*x;}
F<double> x;
x.diff(0, 1);
... f(x).d(0) ...

F<F<double> > f(F<F<double> > x) {return 2*x*x*x;}
F<F<double> > x;
x.diff(0, 1);
x.diff(0, 1).diff(0,1);
... f(x).d(0).d(0) ...

template <typename T>
T f(T x) {return 2*x*x*x;}
T x;
```

Slow and inconvenient

Implementation of Reverse Mode by Overloading

```
(define-structure tape value operation argments)

(set! original+ +)

(define (+ x y)
 (if (tape? x)
     (tape (+ (value x) (value y))
           '+
           (list (arguments x) (arguments y)))
     (original+ x y)))
```

Reverse Mode

$$
\begin{aligned}
x_1 &= f_1(x_0) \\
&\vdots \\
x_n &= f_n(x_{n-1}) \\
\grave{x}_{n-1} &= \mathcal{J}(f_n)(x_{n-1}) \times \grave{x}_n \\
&\vdots \\
\grave{x}_0 &= \mathcal{J}(f_1)(x_0) \times \grave{x}_1
\end{aligned}
$$

Implementation of Reverse Mode by Transformation—I

```
subroutine sqr(x, y)
  y = x * x
  end

subroutine l2(x1, y1, x2, y2, r)
  t1 = x2 - x1
  sqr(t1, t2)
  t3 = y2 - y1
  sqr(t3, t4)
  r = t2 + t4
  end
```

Implementation of Reverse Mode by Transformation—II

```
subroutine sqrf(xp, yp)
  push(xp)
  yp = xp * xp
  end

subroutine l2f(x1p, y1p, x2p, y2p, rp)
  t1p = x2p - x1p
  sqr(t1p, t2p)
  t3p = y2p - y1p
  sqr(t3p, t4p)
  rp = t2p + t4p
  end
```

Implementation of Reverse Mode by Transformation—III

```
subroutine sqrr(xc, yc)
  pop(xp)
  xc = yc * xp
  xc += xp * yc
  end

subroutine l2r(x1c, y1c, x2c, y2c, rc)
  t2c = rc
  t4c = rc
  sqrr(t3c, t4c)
  y2c = -t3c
  y1c = t3c
  sqrr(t1c, t2c)
  x2c = -t1c
  x1c = t1c
  end
```

Key Idea

Migrate reflective source-to-source transformation
from run time to compile time
with abstract interpretation

Traditional AD by Source-to-Source Transformation

Preprocessor at Compile Time

```
function g(x)
   return x+1
end
```

```
function f(x)
   return 2*g(x)
end
```

```
... derivative(f, 3) ...
```

Traditional AD by Source-to-Source Transformation

Preprocessor at Compile Time

```
function g(x)
   return x+1
end

function f(x)
   return 2*g(x)
end

local y, y_tangent = f_forward(3, 1)
... y_tangent ...
```

Traditional AD by Source-to-Source Transformation

Preprocessor at Compile Time

```
function g(x)
   return x+1
end


function f_forward(x, x_tangent)
   local y, y_tangent = g_forward(x, x_tangent)
   return return 2*y, 2*y_tangent
end

local y, y_tangent = f_forward(3, 1)
... y_tangent ...
```

Traditional AD by Source-to-Source Transformation

Preprocessor at Compile Time

```
function g_forward(x, x_tangent)
   local y, y_tangent = x, x_tangent
   return x+1, x_tangent
end

function f_forward(x, x_tangent)
   local y, y_tangent = g_forward(x, x_tangent)
   return return 2*y, 2*y_tangent
end

local y, y_tangent = f_forward(3, 1)
... y_tangent ...
```

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f)
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"

transform("function f(x)
              return 2*g(x)
           end")
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"

transform("function f(x)
              return 2*g(x)
           end") ==> "function f_forward(x, x_tangent)
                         local y, y_tangent = g_forward(x, x_tangent)
                         return return 2*y, 2*y_tangent
                      end"
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"

transform("function f(x)
              return 2*g(x)
           end") ==> "function f_forward(x, x_tangent)
                         local y, y_tangent = g_forward(x, x_tangent)
                         return return 2*y, 2*y_tangent
                      end"

compile("function f_forward(x, x_tangent)
            local y, y_tangent = g_forward(x, x_tangent)
            return return 2*y, 2*y_tangent
         end")
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"

transform("function f(x)
              return 2*g(x)
           end") ==> "function f_forward(x, x_tangent)
                         local y, y_tangent = g_forward(x, x_tangent)
                         return return 2*y, 2*y_tangent
                      end"

compile("function f_forward(x, x_tangent)
            local y, y_tangent = g_forward(x, x_tangent)
            return return 2*y, 2*y_tangent
         end") ==> f_forward
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"

transform("function f(x)
              return 2*g(x)
           end") ==> "function f_forward(x, x_tangent)
                         local y, y_tangent = g_forward(x, x_tangent)
                         return return 2*y, 2*y_tangent
                      end"

compile("function f_forward(x, x_tangent)
            local y, y_tangent = g_forward(x, x_tangent)
            return return 2*y, 2*y_tangent
         end") ==> f_forward

called_by(f)
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"

transform("function f(x)
              return 2*g(x)
           end") ==> "function f_forward(x, x_tangent)
                         local y, y_tangent = g_forward(x, x_tangent)
                         return return 2*y, 2*y_tangent
                      end"

compile("function f_forward(x, x_tangent)
            local y, y_tangent = g_forward(x, x_tangent)
            return return 2*y, 2*y_tangent
         end") ==> f_forward

called_by(f) ==> {g}
```

--

Source-to-Source Transformation at Run Time

Reflection

```
function f(x)
   return 2*g(x)
end

code(f) ==> "function f(x)
                return 2*g(x)
             end"

transform("function f(x)
              return 2*g(x)
           end") ==> "function f_forward(x, x_tangent)
                         local y, y_tangent = g_forward(x, x_tangent)
                         return return 2*y, 2*y_tangent
                      end"

compile("function f_forward(x, x_tangent)
            local y, y_tangent = g_forward(x, x_tangent)
            return return 2*y, 2*y_tangent
         end") ==> f_forward

called_by(f) ==> {g}

function derivative(f, x)
   for g in called_by(f) do compile(transform(code(g))) end
   local y, y_tangent = compile(transform(code(f)))(x, 1)
   return y_tangent
end
--
```

But How Can We Make This Efficient?

```
while not converged() do
   x = x-eta*derivative(f, x)
end
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add(x, y)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = 3, y = 4
... add(x, y) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add(x, y)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = DOUBLE, y = DOUBLE
... add(x, y) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add(x, y)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = DOUBLE, y = DOUBLE
... add(DOUBLE, DOUBLE) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_1(DOUBLE, DOUBLE)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = DOUBLE, y = DOUBLE
... add_1(DOUBLE, DOUBLE) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_1(DOUBLE, DOUBLE)
   if DOUBLE=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = DOUBLE, y = DOUBLE
... add_1(DOUBLE, DOUBLE) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_1(DOUBLE, DOUBLE)
   if false then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = DOUBLE, y = DOUBLE
... add_1(DOUBLE, DOUBLE) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_1(DOUBLE, DOUBLE)



      return scalar_add(x, y)

end

local x = DOUBLE, y = DOUBLE
... add_1(DOUBLE, DOUBLE) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_1(DOUBLE, DOUBLE)



      return scalar_add(x, y)

end

local x = 3, y = 4
... scalar_add(x, y) ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_1(DOUBLE, DOUBLE)



      return scalar_add(x, y)

end

local x = 3, y = 4
... x+y ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add(x, y)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = 3, y = 4
... x+y ...
local x = ARRAY, y = ARRAY
... add(x, y) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add(x, y)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = 3, y = 4
... x+y ...
local x = ARRAY, y = ARRAY
... add(ARRAY, ARRAY) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_2(ARRAY, ARRAY)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = 3, y = 4
... x+y ...
local x = ARRAY, y = ARRAY
... add_2(ARRAY, ARRAY) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_2(ARRAY, ARRAY)
   if ARRAY=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = 3, y = 4
... x+y ...
local x = ARRAY, y = ARRAY
... add_2(ARRAY, ARRAY) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_2(ARRAY, ARRAY)
   if true then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = 3, y = 4
... x+y ...
local x = ARRAY, y = ARRAY
... add_2(ARRAY, ARRAY) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add_2(ARRAY, ARRAY)

      return vector_add(x, y)



end

local x = 3, y = 4
... x+y ...
local x = ARRAY, y = ARRAY
... add_2(ARRAY, ARRAY) ...
```

Abstract Interpretation aka (Polyvariant) Flow Analysis

```
function scalar_add(x, y)
   return x+y
end

function vector_add(x, y)
   local n = x:size(1)
   local z = torch.Tensor(n)
   for i = 1, n do
      z[i] = x[i]+y[i]
   end
   return z
end

function add(x, y)
   if x:type()=="torch.Tensor" then
      return vector_add(x, y)
   else
      return scalar_add(x, y)
   end
end

local x = 3, y = 4
... x+y ...
local x = torch.Tensor(5):zeros(), y = torch.Tensor(5):zeros()
... vector_add(x, y) ...
```

A Single Powerful Optimization

```
{x = e1, y = e2}.x
```

A Single Powerful Optimization

`{x = e1, y = e2}.x` $\leadsto$ `e1`

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- can eliminate storage allocation

A Single Powerful Optimization

`{x = e1, y = e2}.x` $\rightsquigarrow$ `e1`

- can eliminate storage allocation
- can eliminate storage reclamation

A Single Powerful Optimization

$$\texttt{\{x = e1, y = e2\}.x} \rightsquigarrow \texttt{e1}$$

- can eliminate storage allocation
- can eliminate storage reclamation
- can eliminate storage writes

A Single Powerful Optimization

$$\texttt{\{x = e1, y = e2\}.x} \leadsto \texttt{e1}$$

- can eliminate storage allocation
- can eliminate storage reclamation
- can eliminate storage writes
- can eliminate storage reads

A Single Powerful Optimization

```
{x = e1, y = e2}.x ⇝ e1
```

- can eliminate storage allocation
- can eliminate storage reclamation
- can eliminate storage writes
- can eliminate storage reads
- can eliminate dead code

The Kind of Code People Write in Dynamic Languages

```
function map(f, x)
   y = torch.Tensor(x:size(1))
   for i = 1, x:size(1) do
      y[i] = f(x[i])
   end
   return y
end

function reduce(g, i, x)
   y = i
   for i = 1, x:size(1) do
      y = g(y, x[i])
   end
   return y
end

reduce(function(x, y) return x+y end,
       0,
       map(function(x) return x*x end, torch.Tensor({u, v, w, x, y})))
```

--

The Kind of Code People Write in Dynamic Languages

```
function map(f, x)
   y = torch.Tensor(x:size(1))
   for i = 1, x:size(1) do
      y[i] = f(x[i])
   end
   return y
end

function reduce(g, i, x)
   y = i
   for i = 1, x:size(1) do
      y = g(y, x[i])
   end
   return y
end

reduce(function(x, y) return x+y end,
       0,
       map(function(x) return x*x end, torch.Tensor({u, v, w, x, y})))

u*u + v*v + w*w + x*x + y*y

--
```

Key Idea

You need this anyway
to compile dynamic languages efficiently

Key Idea

Same mechanism can support AD

Migrating Reflective AD from Run Time to Compile Time

```
function f(x)
   return 2*x
end
```

```
function derivative(g, x)
   local y, y_tangent = compile(transform(code(g)))(x, 1)



   return y_tangent
end
```

```
... derivative(f, 3) ...
```

Migrating Reflective AD from Run Time to Compile Time

```
function f(x)
   return 2*x
end
```

```
function derivative_1(g, x)
   local y, y_tangent = compile(transform(code(g)))(x, 1)



   return y_tangent
end
```

```
... derivative_1(FUNCTION_F, 3) ...
```

Migrating Reflective AD from Run Time to Compile Time

```
function f(x)
   return 2*x
end
```

```
function derivative_1(FUNCTION_F, x)
   local y, y_tangent = compile(transform(code(FUNCTION_F)))(x, 1)



   return y_tangent
end
```

```
... derivative_1(FUNCTION_F, 3) ...
```

Migrating Reflective AD from Run Time to Compile Time

```
function f(x)
   return 2*x
end
```

```
function derivative_1(FUNCTION_F, x)
   local y, y_tangent = compile(transform("function f(x)
                                              return 2*x
                                           end"))(x, 1)

   return y_tangent
end
```

```
... derivative_1(FUNCTION_F, 3) ...
```

Migrating Reflective AD from Run Time to Compile Time

```
function f(x)
   return 2*x
end
```

```
function derivative_1(FUNCTION_F, x)
   local y, y_tangent = compile("function f_forward(x, x_tangent)
                                    local y, y_tangent = 2*x, 2*x_tangent
                                    return y, y_tangent
                                 end")(x, 1)
   return y_tangent
end
```

```
... derivative_1(FUNCTION_F, 3) ...
```

Migrating Reflective AD from Run Time to Compile Time

```
function f(x)
   return 2*x
end

function f_forward(x, x_tangent)
   local y, y_tangent = 2*x, 2*x_tangent
   return y, y_tangent
end

function derivative_1(FUNCTION_F, x)
   local y, y_tangent = f_forward(x, 1)



   return y_tangent
end


... derivative_1(FUNCTION_F, 3) ...
```

Migrating Reflective AD from Run Time to Compile Time

```
function f(x)
   return 2*x
end

function f_forward(x, x_tangent)
   local y, y_tangent = 2*x, 2*x_tangent
   return y, y_tangent
end

function derivative(g, x)
   local y, y_tangent = compile(transform(code(g)))(x, 1)



   return y_tangent
end

local y, y_tangent = f_forward(x, 1)
... y_tangent ...
```

A Single Powerful Optimization

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- separates AD from optimization

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- allows simple formulation of AD transforms

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  (forward mode is 28 lines; reverse mode is 155 lines)

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A Single Powerful Optimization

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- many AD optimizations (like TBR) fall out
- makes it easier to get it right

A Single Powerful Optimization

- separates AD from optimization
- allows simple formulation of AD transforms
  (forward mode is 28 lines; reverse mode is 155 lines)
- tape is a data structure (in the language)
- many AD optimizations (like TBR) fall out
- makes it easier to get it right
- makes it easier to get it to nest

Essence of Forward Transform

$$
\begin{array}{rcl}
\overrightarrow{c} & \leadsto & \overrightarrow{\mathcal{J}}\; c \\
\overrightarrow{\lambda x.e} & \leadsto & \lambda\,\overrightarrow{x}\,.\,\overrightarrow{e} \\
\overrightarrow{e_1\; e_2} & \leadsto & \overrightarrow{e_1}\;\overrightarrow{e_2} \\
\overrightarrow{\textbf{letrec}\; x_1 = e_1; \ldots; x_n = e_n\; \textbf{in}\; e} & \leadsto & \textbf{letrec}\;\overrightarrow{x_1} = \overrightarrow{e_1}; \ldots; \overrightarrow{x_n} = \overrightarrow{e_n}\;\textbf{in}\;\overrightarrow{e} \\
\overrightarrow{e_1, e_2} & \leadsto & \overrightarrow{e_1}\,\overrightarrow{,}\,\overrightarrow{e_2}
\end{array}
$$

Essence of Reverse Transform

$$
\begin{aligned}
\overleftarrow{x = c} &\rightsquigarrow \overleftarrow{x} = \overleftarrow{\mathcal{J}}\, c \\
\overleftarrow{x_1 = x_2} &\rightsquigarrow \overleftarrow{x_1} = \overleftarrow{x_2} \\
\overleftarrow{x = \lambda x.e} &\rightsquigarrow \overleftarrow{x} = \overleftarrow{\lambda x.e} \\
\overleftarrow{x = x_1\ x_2} &\rightsquigarrow \overleftarrow{x}, \bar{x} = \overleftarrow{x_1}\ \overleftarrow{x_2} \\
\overleftarrow{x = x_1, x_2} &\rightsquigarrow \overleftarrow{x} = \overleftarrow{x_1} \overleftarrow{,} \overleftarrow{x_2}
\end{aligned}
$$

$$
\begin{aligned}
\overline{x_1 = x_2} &\rightsquigarrow \overleftharpoon{x_2} += \overleftharpoon{x_1} \\
\overline{x = \lambda x.e} &\rightsquigarrow \overleftharpoon{\lambda x.e} += \overleftharpoon{x} \\
\overline{x = x_1\ x_2} &\rightsquigarrow \overleftharpoon{x_1}\ \overleftharpoon{,}\ \overleftharpoon{x_2} += \bar{x}\ \overleftharpoon{x} \\
\overline{x = x_1, x_2} &\rightsquigarrow \overleftharpoon{x_1}\ \overleftharpoon{,}\ \overleftharpoon{x_2} += \overleftharpoon{x}
\end{aligned}
$$

$$
\overleftarrow{\lambda x.\textbf{let } b_1; \ldots; b_n \textbf{ in } y} \rightsquigarrow \lambda \overleftarrow{x}.\textbf{let } \overleftarrow{b_1}; \ldots; \overleftarrow{b_n} \textbf{ in } \overleftarrow{y}, \lambda \overleftharpoon{y}.\textbf{let } \overline{b_n}; \ldots; \overline{b_1} \textbf{ in } \overleftharpoon{x}
$$

Game Theory

				B		
		b_1	$\dots$	b_j	$\dots$	b_n
	a_1					
	$\vdots$		$\ddots$	$\vdots$		
A	a_i		$\dots$	$\textsc{Payoff}(a_i, b_j)$	$\dots$	
	$\vdots$			$\vdots$	$\ddots$	
	a_m					

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

Game Theory

$$
\begin{array}{cc|ccccc}
 & & & & B & & \\
 & & b_1 & \dots & b_j & \dots & b_n \\
\hline
 & a_1 & & & & & \\
 & \vdots & & \ddots & \vdots & & \\
A & a_i & & \dots & \textsc{Payoff}(a_i, b_j) & \dots & \\
 & \vdots & & & \vdots & \ddots & \\
 & a_m & & & & &
\end{array}
$$

$$\max_{a \in A} \min_{b \in B} \textsc{Payoff}(a, b)$$

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

Game Theory

$$
\begin{array}{cc|ccc}
 & & & \mathbb{R}^n & \\
 & & \dots & \mathbf{b} & \dots \\
\hline
 & \vdots & \ddots & \vdots & \\
\mathbb{R}^m & \mathbf{a} & \dots & \textsc{Payoff}(\mathbf{a}, \mathbf{b}) & \dots \\
 & \vdots & & \vdots & \ddots
\end{array}
$$

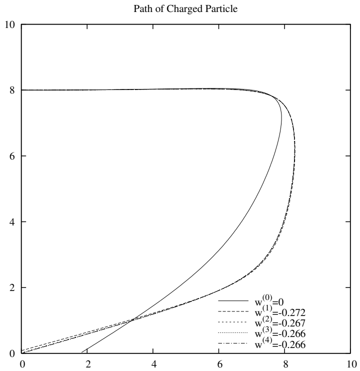
$$
\max_{\mathbf{a} \in \mathbb{R}^m} \min_{\mathbf{b} \in \mathbb{R}^n} \textsc{Payoff}(\mathbf{a}, \mathbf{b})
$$

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.

Code

```
(letrec ((loop
          (lambda (i r)
           (if (zero? i)
               r
               (loop (- i 1)
                     (let* ((start (list (real 1) (real 1)))
                            (f (lambda (x1 y1 x2 y2)
                                (- (+ (sqr x1) (sqr y1))
                                   (+ (sqr x2) (sqr y2)))))
                            ((list x1* y1*)
                             (multivariate-argmin-F
                              (lambda ((list x1 y1))
                               (multivariate-max-F
                                (lambda ((list x2 y2)) (f x1 y1 x2 y2))
                                start))
                              start))
                            ((list x2* y2*)
                             (multivariate-argmax-F
                              (lambda ((list x2 y2)) (f x1* y1* x2 y2))
                              start)))
                      (list (list (write-real x1*) (write-real y1*))
                            (list (write-real x2*) (write-real y2*)))))))))
 (loop (real 1000) (list (list (real 0) (real 0)) (list (real 0) (real 0)))))
```

Cathode Ray Tubes



$$
\begin{aligned}
\text{potential: } p(\mathbf{x};w) &= \|\mathbf{x}-(10,10-w)\|^{-1} + \|\mathbf{x}-(10,0)\|^{-1} \\
\ddot{\mathbf{x}}(t) &= -\nabla_{\mathbf{x}}\, p(\mathbf{x})|_{\mathbf{x}=\mathbf{x}(t)} \\
\dot{\mathbf{x}}(t+\Delta t) &= \dot{\mathbf{x}}(t) + \Delta t\, \ddot{\mathbf{x}}(t) \\
\mathbf{x}(t+\Delta t) &= \mathbf{x}(t) + \Delta t\, \dot{\mathbf{x}}(t) \\
\text{When: } x_1(t+\Delta t) &\le 0 \\
\text{let: } \Delta t_f &= -x_1(t)/\dot{x}_1(t) \\
t_f &= t + \Delta t_f \\
\mathbf{x}(t_f) &= \mathbf{x}(t) + \Delta t_f\, \dot{\mathbf{x}}(t) \\
\text{Error: } E(w) &= x_0(t_f)^2 \\
\text{Find: } & \operatorname*{argmin}_w E(w)
\end{aligned}
$$

Sprague, C. S. and George, R. H. (1939). *Cathode Ray Deflecting Electrode*. US Patent 2,161,437.

George, R. H. (1940). *Cathode Ray Tube*. US Patent 2,222,942.

Code

```
(define (naive-euler w)
 (let* ((charges
         (list (list (real 10) (- (real 10) w)) (list (real 10) (real 0))))
        (x-initial (list (real 0) (real 8)))
        (xdot-initial (list (real 0.75) (real 0)))
        (delta-t (real 1e-1))
        (p (lambda (x)
            ((reduce + (real 0))
             ((map (lambda (c) (/ (real 1) (distance x c)))) charges)))))
  (letrec ((loop (lambda (x xdot)
                  (let* ((xddot (k*v (real -1) ((gradient-F p) x)))
                         (x-new (v+ x (k*v delta-t xdot))))
                   (if (positive? (list-ref x-new 1))
                       (loop x-new (v+ xdot (k*v delta-t xddot)))
                       (let* ((delta-t-f (/ (- (real 0) (list-ref x 1))
                                            (list-ref xdot 1)))
                              (x-t-f (v+ x (k*v delta-t-f xdot))))
                        (sqr (list-ref x-t-f 0))))))))
   (loop x-initial xdot-initial))))

(letrec ((loop
          (lambda (i r)
           (if (zero? i)
               r
               (loop (- i 1)
                     (let* ((w0 (real 0))
                            ((list w*)
                             (multivariate-argmin-F
                              (lambda ((list w)) (naive-euler w)) (list w0))))
                      (write-real w*)))))))
 (loop (real 1000) (real 0)))
```

Probabilistic Lambda Calculus

$P = \textbf{if } x_0 \textbf{ then } 0 \textbf{ else if } x_1 \textbf{ then } 1 \textbf{ else } 2$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Lambda Calculus

P = **if** x_0 **then** 0 **else if** x_1 **then** 1 **else** 2

$\Pr(x_0 \mapsto \textbf{true}) = p_0$ $\Pr(x_0 \mapsto \textbf{false}) = 1 - p_0$
$\Pr(x_1 \mapsto \textbf{true}) = p_1$ $\Pr(x_1 \mapsto \textbf{false}) = 1 - p_1$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

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$\Pr(x_1 \mapsto \textbf{true}) = p_1$ $\Pr(x_1 \mapsto \textbf{false}) = 1 - p_1$

$\Pr(\mathcal{E}(P) = 0 | p_0, p_1) = p_0$
$\Pr(\mathcal{E}(P) = 1 | p_0, p_1) = (1 - p_0)p_1$
$\Pr(\mathcal{E}(P) = 2 | p_0, p_1) = (1 - p_0)(1 - p_1)$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Lambda Calculus

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$\Pr(x_0 \mapsto \textbf{true}) = p_0$ $\qquad\qquad$ $\Pr(x_0 \mapsto \textbf{false}) = 1 - p_0$
$\Pr(x_1 \mapsto \textbf{true}) = p_1$ $\qquad\qquad$ $\Pr(x_1 \mapsto \textbf{false}) = 1 - p_1$

$$\Pr(\mathcal{E}(P) = 0 | p_0, p_1) = p_0$$
$$\Pr(\mathcal{E}(P) = 1 | p_0, p_1) = (1 - p_0)p_1$$
$$\Pr(\mathcal{E}(P) = 2 | p_0, p_1) = (1 - p_0)(1 - p_1)$$

$$\prod_{v \in \{0,1,2,2\}} \Pr(\mathcal{E}(P) = v | p_0, p_1) = p_0(1 - p_0)^3 p_1 (1 - p_1)^2$$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Lambda Calculus

$P = \textbf{if } x_0 \textbf{ then } 0 \textbf{ else if } x_1 \textbf{ then } 1 \textbf{ else } 2$

$\Pr(x_0 \mapsto \textbf{true}) = p_0$ $\quad\quad\quad$ $\Pr(x_0 \mapsto \textbf{false}) = 1 - p_0$

$\Pr(x_1 \mapsto \textbf{true}) = p_1$ $\quad\quad\quad$ $\Pr(x_1 \mapsto \textbf{false}) = 1 - p_1$

$$\Pr(\mathcal{E}(P) = 0|p_0, p_1) = p_0$$
$$\Pr(\mathcal{E}(P) = 1|p_0, p_1) = (1 - p_0)p_1$$
$$\Pr(\mathcal{E}(P) = 2|p_0, p_1) = (1 - p_0)(1 - p_1)$$

$$\prod_{v \in \{0,1,2,2\}} \Pr(\mathcal{E}(P) = v|p_0, p_1) = p_0(1 - p_0)^3 p_1(1 - p_1)^2$$

$$\operatorname*{argmax}_{p_0, p_1} \prod_{v \in \{0,1,2,2\}} \Pr(\mathcal{E}(P) = v|p_0, p_1) = \left\langle \frac{1}{4}, \frac{1}{3} \right\rangle$$

Koller, D., McAllester, D. , and Pfeffer, A. (1997). *Effective Bayesian Inference for Stochastic Programs*. Proceedings of the 14th National Conference on Artificial Intelligence (AAAI), pp. 740–7.

Probabilistic Prolog

```
p(0).
p(X):-q(X).
q(1).
q(2).
```

Probabilistic Prolog

$\Pr(\texttt{p(0).}) = p_0$
$\Pr(\texttt{p(X):-q(X).}) = 1 - p_0$
$\Pr(\texttt{q(1).}) = p_1$
$\Pr(\texttt{q(2).}) = 1 - p_1$

Probabilistic Prolog

$\Pr(\texttt{p(0).}) = p_0$
$\Pr(\texttt{p(X):-q(X).}) = 1 - p_0$
$\Pr(\texttt{q(1).}) = p_1$
$\Pr(\texttt{q(2).}) = 1 - p_1$

$\Pr(\texttt{?-p(0).}) = p_0$
$\Pr(\texttt{?-p(1).}) = (1 - p_0)p_1$
$\Pr(\texttt{?-p(2).}) = (1 - p_0)(1 - p_1)$

Probabilistic Prolog

$$\Pr(\texttt{p(0).}) = p_0$$
$$\Pr(\texttt{p(X):-q(X).}) = 1 - p_0$$
$$\Pr(\texttt{q(1).}) = p_1$$
$$\Pr(\texttt{q(2).}) = 1 - p_1$$

$$\Pr(\texttt{?-p(0).}) = p_0$$
$$\Pr(\texttt{?-p(1).}) = (1 - p_0)p_1$$
$$\Pr(\texttt{?-p(2).}) = (1 - p_0)(1 - p_1)$$

$$\prod_{q \in \{\texttt{p(0)},\texttt{p(1)},\texttt{p(2)},\texttt{p(2)}\}} \Pr(\texttt{?-}q\texttt{.}) = p_0(1 - p_0)^3 p_1(1 - p_1)^2$$

Probabilistic Prolog

$$\Pr(\texttt{p(0).}) = p_0$$
$$\Pr(\texttt{p(X):-q(X).}) = 1 - p_0$$
$$\Pr(\texttt{q(1).}) = p_1$$
$$\Pr(\texttt{q(2).}) = 1 - p_1$$

$$\Pr(\texttt{?-p(0).}) = p_0$$
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$$\operatorname*{argmax}_{p_0,p_1} \prod_{q \in \{\texttt{p(0)},\texttt{p(1)},\texttt{p(2)},\texttt{p(2)}\}} \Pr(\texttt{?-}q\texttt{.}) = \left\langle \frac{1}{4}, \frac{1}{3} \right\rangle$$

Probabilistic Lambda Calculus

```
(define (evaluate expression environment)
 (cond
  ((constant-expression? expression)
   (singleton-tagged-distribution
    (constant-expression-value expression)))
  ((variable-access-expression? expression)
   (lookup-value
    (variable-access-expression-variable expression) environment))
  ((lambda-expression? expression)
   (singleton-tagged-distribution
    (lambda (tagged-distribution)
     (evaluate
      (lambda-expression-body expression)
      (cons (make-binding (lambda-expression-variable expression)
                          tagged-distribution)
            environment)))))
  (else (let ((tagged-distribution
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                         environment)))
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Probabilistic Lambda Calculus

```
(gradient-ascent
 (lambda (p)
  (let ((tagged-distribution
         (evaluate if x0 then 0 else if x1 then 1 else 2
                   (list Pr(x0 ↦ true) = p0   Pr(x0 ↦ false) = 1 − p0
                         Pr(x1 ↦ true) = p1   Pr(x1 ↦ false) = 1 − p1
                         ...)))))
   (map-reduce
    *
    1.0
    (lambda (value)
     (likelihood value tagged-distribution))
    '(0 1 2 2))))
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Probabilistic Prolog

```
(define (proof-distribution term clauses)
 (let ((offset ...))
  (map-reduce
   append
   '()
   (lambda (clause)
    (let ((clause (alpha-rename clause offset)))
     (let loop ((p (clause-p clause))
                (substitution (unify term (clause-term clause)))
                (terms (clause-terms clause)))
      (if (boolean? substitution)
          '()
          (if (null? terms)
              (list (make-double p substitution))
              (map-reduce
               append
               '()
               (lambda (double)
                (loop (* p (double-p double))
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               (proof-distribution
                (apply-substitution substitution (first terms)) clauses)))))))
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Probabilistic Prolog

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  (let ((clauses (list Pr(p(0).) = p0
                       Pr(p(X):-q(X).) = 1 - p0
                       Pr(q(1).) = p1
                       Pr(q(2).) = 1 - p1)))
   (map-reduce
    *
    1.0
    (lambda (query)
     (likelihood (proof-distribution query clauses)))
    '(p(0) p(1) p(2) p(2)))))
 '(0.5 0.5)
 1000.0
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Probabilistic Prolog

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(gradient-ascent
 (lambda (p)
  (let ((clauses (list Pr(p(0).) = p0
                       Pr(p(X):-q(X).) = 1 - p0
                       Pr(q(1).) = p1
                       Pr(q(2).) = 1 - p1)))
   (map-reduce
    *
    1.0
    (lambda (query)
     (likelihood (proof-distribution query clauses)))
    '(p(0) p(1) p(2) p(2)))))
 '(0.5 0.5)
 1000.0
 0.1)
```

Probabilistic Prolog

```
(gradient-ascent
 (lambda (p)
  (let ((clauses (list Pr(p(0).) = p0
                       Pr(p(X):-q(X).) = 1 - p0
                       Pr(q(1).) = p1
                       Pr(q(2).) = 1 - p1)))
   (map-reduce
    *
    1.0
    (lambda (query)
     (likelihood (proof-distribution query clauses)))
    '(p(0) p(1) p(2) p(2)))))
 '(0.5 0.5)
 1000.0
 0.1)
```

Generated Code

```
static void f2679(double a_f2679_0,double a_f2679_1,double a_f2679_2,double a_f2679_3){
  int t272381=((a_f2679_2==0.)?0:1);
  double t272406;
  double t272405;
  double t272404;
  double t272403;
  double t272402;
  if((t272381==0)){
    double t272480=(1.-a_f2679_0);
    double t272572=(1.-a_f2679_1);
    double t273043=(a_f2679_0+0.);
    double t274185=(t272480*a_f2679_1);
    double t274426=(t274185+0.);
    double t275653=(t272480*t272572);
    double t275894=(t275653+0.);
    double t277121=(t272480*t272572);
    double t277362=(t277121+0.);
    double t277431=(t277362*1.);
    double t277436=(t275894*t277431);
    double t277441=(t274426*t277436);
    double t277446=(t273043*t277441);
    ...
    double t1777107=(t1774696+t1715394);
    double t1777194=(0.-t1745420);
    double t1778533=(t1777194+t1419700);
    t272406=a_f2679_0;
    t272405=a_f2679_1;
    t272404=t277446;
    t272403=t1778533;
    t272402=t1777107;}
  else {...}
  r_f2679_0=t272406;
  r_f2679_1=t272405;
  r_f2679_2=t272404;
  r_f2679_3=t272403;
  r_f2679_4=t272402;}
```

Benchmarks

		backprop		
		Fs	Fv	R
VLAD	STALIN∇	1.00	■	1.00
FORTRAN	ADIFOR	15.51	3.35	■
	TAPENADE	14.97	5.97	6.86
C	ADIC	22.75	5.61	■
C++	ADOL–C	12.16	5.79	32.77
	CPPAD	54.74	■	29.24
	FADBAD++	132.31	46.01	60.71
ML	MLTON	95.20	■	39.90
	OCAML	202.01	■	156.93
	SML/NJ	181.93	■	102.89
HASKELL	GHC	■	■	■
SCHEME	BIGLOO	743.26	■	360.07
	CHICKEN	1626.73	■	1125.24
	GAMBIT	671.54	■	379.63
	IKARUS	279.59	■	165.16
	LARCENY	1203.34	■	511.54
	MIT SCHEME	2446.33	■	1113.09
	MZC	1318.60	■	754.47
	MZSCHEME	1364.14	■	772.10
	SCHEME->C	597.67	■	280.93
	SCMUTILS	5889.26	■	■
	STALIN	435.82	■	281.27

Damned Benchmarks

		particle				saddle			
		FF	FR	RF	RR	FF	FR	RF	RR
VLAD	STALIN∇	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
FORTRAN	ADIFOR	2.05	🟦	🟦	🟦	5.44	🟦	🟦	🟦
	TAPENADE	5.51	🟥	🟩	🟥	8.09	🟥	🟩	🟥
C	ADIC	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥
C++	ADOL–C	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥
	CPPAD	🟥	🟥	🟥	🟥	🟥	🟥	🟥	🟥
	FADBAD++	93.32	🟩	🟩	🟩	60.67	🟩	🟩	🟩
ML	MLTON	78.13	111.27	45.95	32.57	114.07	146.28	12.27	10.58
	OCAML	217.03	415.64	352.06	261.38	291.26	407.67	42.39	50.21
	SML/NJ	153.01	226.84	270.63	192.13	271.84	299.76	25.66	23.89
HASKELL	GHC	209.44	🟩	🟩	🟩	247.57	🟩	🟩	🟩
SCHEME	BIGLOO	627.78	855.70	275.63	187.39	1004.85	1076.73	105.24	89.23
	CHICKEN	1453.06	2501.07	821.37	1360.00	2276.69	2964.02	225.73	252.87
	GAMBIT	578.94	879.39	356.47	260.98	958.73	1112.70	89.99	89.23
	IKARUS	266.54	386.21	158.63	116.85	424.75	527.57	41.27	42.34
	LARCENY	964.18	1308.68	360.68	272.96	1565.53	1508.39	126.44	112.82
	MIT SCHEME	2025.23	3074.30	790.99	609.63	3501.21	3896.88	315.17	295.67
	MZC	1243.08	1944.00	740.31	557.45	2135.92	2434.05	194.49	187.53
	MZSCHEME	1309.82	1926.77	712.97	555.28	2371.35	2690.64	224.61	219.29
	SCHEME->C	582.20	743.00	270.83	208.38	910.19	913.66	82.93	69.87
	SCMUTILS	4462.83	🟦	🟦	🟦	7651.69	🟦	🟦	🟦
	STALIN	364.08	547.73	399.39	295.00	543.68	690.64	63.96	52.93

Statistics

		probabilistic-lambda-calculus F	probabilistic-lambda-calculus R	probabilistic-prolog F	probabilistic-prolog R
VLAD	STALIN∇	1.00	1.00	1.00	1.00
FORTRAN	ADIFOR	🟩	🟦	🟩	🟦
	TAPENADE	🟩	🟩	🟩	🟩
C	ADIC	🟩	🟦	🟩	🟦
C++	ADOL–C	🟩	🟩	🟩	🟩
	CPPAD	🟩	🟩	🟩	🟩
	FADBAD++	🟩	🟩	🟩	🟩
ML	MLTON	129.11	114.88	848.45	507.21
	OCAML	249.40	499.43	1260.83	1542.47
	SML/NJ	234.62	258.53	2505.59	1501.17
HASKELL	GHC	🟩	🟩	🟩	🟩
SCHEME	BIGLOO	983.12	1016.50	12832.92	7918.21
	CHICKEN	2324.54	3040.44	44891.04	24634.44
	GAMBIT	1033.46	1107.26	26077.48	14262.70
	IKARUS	497.48	517.89	8474.57	4845.10
	LARCENY	1658.27	1606.44	25411.62	14386.61
	MIT SCHEME	4130.88	3817.57	87772.39	49814.12
	MZC	2294.93	2346.13	57472.76	31784.38
	MZSCHEME	2721.35	2625.21	60269.37	33135.06
	SCHEME->C	811.37	803.22	10605.32	5935.56
	SCMUTILS	7699.14	🟦	83656.17	🟦
	STALIN	956.47	1994.44	15048.42	16939.28

Take-Home Message

Powerful and efficient AD can be attained by:

- integrating AD into compiler
- formulating AD as one of many compiler transformations
- using abstract interpretation to migrate AD transformation from run time to compile time